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Abridged Notation in Analytic Plane Geometry

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Abstract

An examination of the abridged notation that Salmon introduces in his treatment of lines, circles and conics. Explaining what he means by abridged notation, and showing how he uses it to study various loci in plane geometry. Culminating in its use to show how it may prove the theorems of Pascal and Brianchon, the theorem of Steiner on Pascals hexagons and Steiner's solution of Malfatti's problem.

Reference: **George Salmon**: "A Treatise on Conic Sections"
Longman, Brown, Green and Longmans, London 1855.

Introduction

"A Point is that which cannot be divided."

So wrote Euclid more than 2000 years ago and it is still a reasonable definition of a point after all this time. However, it is not the way that we usually define points in geometry today, and similarly Euclid's other axioms have been iterated on over the years. Definitions are a side of mathematics that it is easy to brush past without too much consideration, to see definitions as the dull but necessary steppingstone to more interesting things. However, I hope to demonstrate in this thesis that understanding the consequences of definitions provides a greater understanding of the subtle challenges inherent in any algebraic treatment of geometry.

We will examine a different way of defining and discussing planar geometry algebraically, using something Salmon called "Abridged Notation". We will build up an understanding of this new system of thinking and then apply it to four classical problems to show how it can be used. By the end of this thesis we should understand the change of perspective this notation offers; the advantages and disadvantages compared to the usual Cartesian system and the choices we make by employing a given set of definitions.

Contents

A Note on References

This thesis is heavily based on Salmon's book and, as such, at the beginning of each numbered part a reference appears; showing how the topics of this thesis connect to Salmon's writings. The references may sometimes point to an entire chapter using the form [Salmon Ch.00] or, for more specific references, we use Salmon's item numbers in the form [Salmon 000]. Item numbers are used in place of page numbers because they remain more consistent across different editions of Salmon's book.

The Cartesian Plane

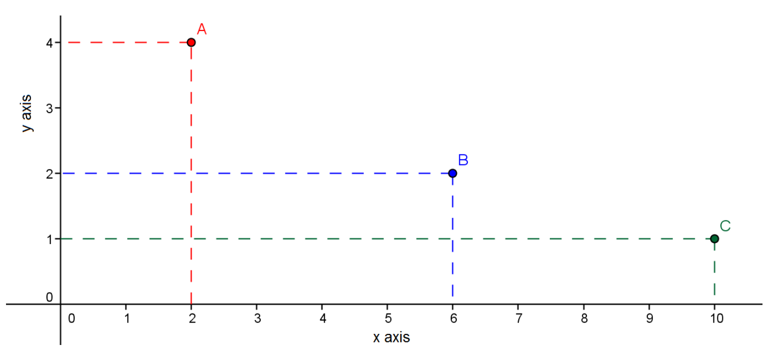
We will begin by quickly establishing what we mean by lines, points, coordinates and some other terms using the common notation of René Descartes. When Decartes first formulated the idea in the 17th century for a method of describing geometric notions with numbers and equations it revolutionized many areas of mathematics. Named after him, the Cartesian coordinate system provides a rigorous way to describe Euclidean geometry in algebraic terms using a pair of oriented, graded axes to give reference to objects in a plane.

To established mathematicians this section should not present anything unexpected, serving as an uncontroversial basis for the greater content of this thesis. [Salmon Ch.1 & Ch.2]

Points and Coordinates

The most basic and fundamental building block of geometry is the point, an exact location. Often the points we are interested in can be described in terms of the property that makes them interesting such as points of intersection, particularly the vertices of shapes which are simply the intersections of the edges. Alternatively, we may choose a point in a plane arbitrarily and ask how it might be described.

In Cartesian coordinates we describe points in a plane using two numbers taken by comparing the point against graded axes that intersect at the *origin*. For each axis we generate an ordinate of the point relative to that axis. Then we combine these numbers into a pair of co-ordinates that precisely and uniquely describe our point.

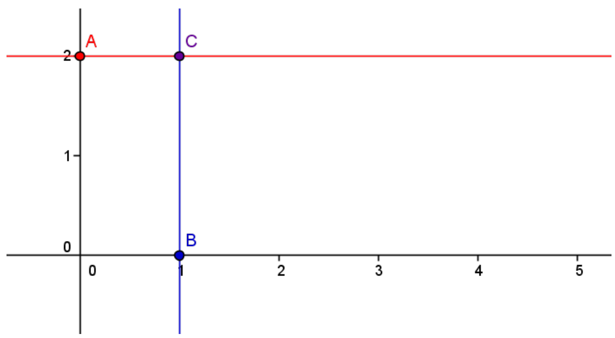


We see above a basic and familiar construction of points against rectangular axes, with the horizontal axis named x and the vertical axis named y. In this construction we can see that A may be represented by (2, 4), B by (6, 2) and C by (10, 1). This is something that we learn in school, a simple way to take a geometric notion and translate it into a numerical form.

Lines and Equations

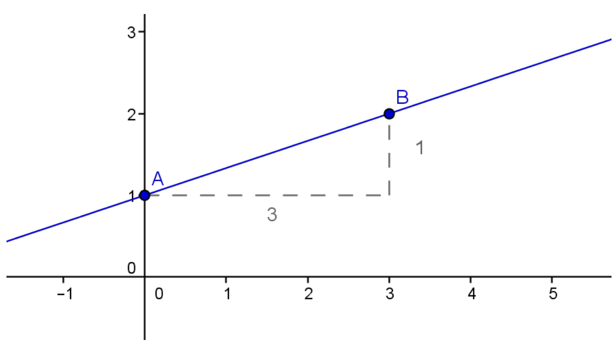
After a point the next concept that must be understood is that of lines, and to understand them we must introduce algebra. Let us imagine an arbitrary point lying somewhere in the plane; then, as we have discussed, it must be represented by two numbers which we shall call x and y. We adopt this naming convention since x here represents the ordinate of the point relative to the x axis and similarly y represents the ordinate relative to the y axis.

If we then fix either x or y at a chosen value the results are shown in the figure.



Consider the blue line represented by $x = 1$, since we have not specified a value for y we can see that the resulting object ranges over all values of y creating a line parallel to the y axis. In this sense our line is simply an infinite collection of points in the form (1, y), all obeying a common rule. In a similar way, the red line is represented by $y = 2$ with x ranging over all possible values to give a line parallel to the x axis.

The examples above are the simplest possible cases for a rule defining a line, simply fixing one coordinate. In order to create lines that are not parallel to an axis we must introduce more complex equations.



To easily construct a general line we require two quantities; a value for y when $x = 0$, and a ratio for the growth of y relative to x. In the above figure we see that when $x = 0$ the line intersects the y axis at $y = 1$ so our first quantity is 1 in this example, we call this value the *constant*. Secondly, when we advance 1 unit in the x direction we advance $\frac{1}{3}$ of a unit in the y direction so our ratio is $\frac{1}{3}$ and we shall call it the *gradient*. We then combine our two quantities into what is called a *linear equation*, taking the form $y = mx + c$, so in the case of our example

$$y = \frac{1}{3}x + 1.$$

The use of the letters m and c to represent gradient and constant terms respectively is a historical convention and one could equally write $y = ax + b$ using a and b, or any other pair of algebraic letters. There are also other equivalent formulations, notably

$$y - mx - c = 0.$$

This form is useful since by presenting the equation equal to zero we may most easily compare and equate it to other lines and objects. Another form is the general linear equation, where A, B and C are constants,

$$Ay + Bx + C = 0.$$

This may be easily transformed into the previous case by simply diving all the terms by A.

Worth noting here is that, in Cartesian coordinates, when we multiply or divide everything in an equation by the same amount we do not change the line that the equation represents. We say that the equation defines the line *up to multiplication by a constant*, precisely meaning this property. This quality is crucial as it allows us to perform algebraic manipulations without losing the meaning of our equations. We also refer to this property as the equations being *normalized*.

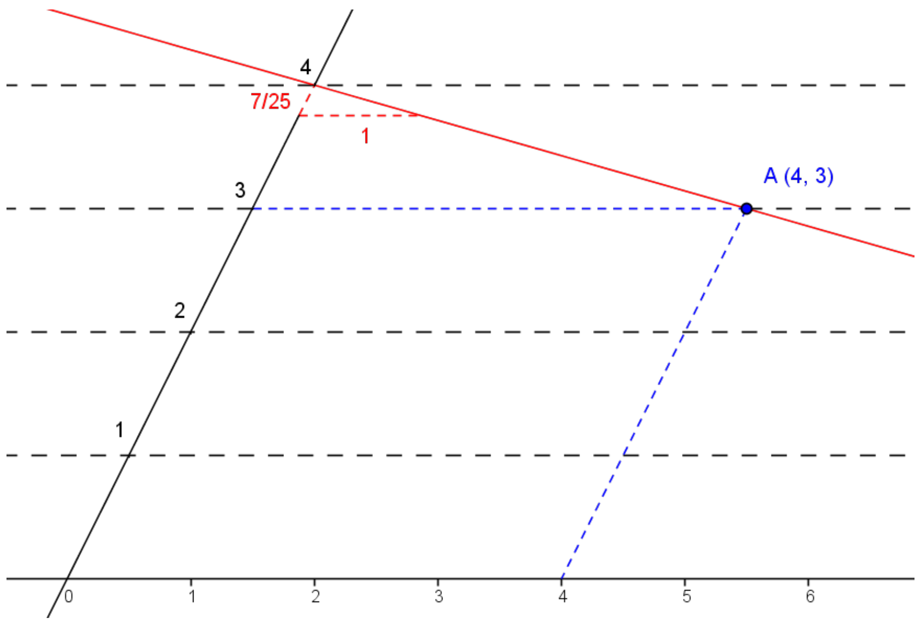
In the course of learning mathematics much study is directed towards linear equations and the lines they represent in the Cartesian coordinate system. Lines showcase clearly how an equation may represent a geometric object and vice versa, the link that makes Cartesian coordinates so powerful. However, we are not overly concerned in the further study of Cartesian lines, we move now to one last consideration before introducing Salmon's abridged notation.

Rectangular and Oblique Axes

It is worth noting that we have, up to now, spoken relatively little about the very foundation of Cartesian coordinates, the axes that we use as references to describe all other objects in the plane. We have said that they exist and that they are oriented and graded but the concept deserves a further treatment.

The axes are, naively, two lines that meet in a single finite point. It is important that they do not meet at infinity or at more than one point as this would make them parallel or coincident, rendering them useless as axes. We then define the intersection point to be the *origin* with the coordinates (0, 0) and we grade both axes, marking off each unit of distance. Lastly we choose an *orientation* which simply defines which side of the axes correspond to positive numbers leaving the other to correspond to the negative numbers. The orientation is also sometimes intuitively called the *direction* that the axes point.

An important thing to note is that there is no requirement that these axes are at right angles to one another. If the axes are at ninety degrees, right angled, to one another we call them rectangular, a pair of rectangular axes, if they form any other angle with each other then we call them oblique. If the axes are oblique we can apply the same methods as for rectangular axes to define the coordinates of points and the equations of lines



The figure is an example of a pair of oblique axes with the point $A(4,\ 3)$ shown in blue, lying on the red line $y = 4 - \frac{7}{25}x$.

This demonstrates that everything we have shown up to now for rectangular axes is equally true in an oblique setting. Rectangular axes are simply a special case, forming a specific angle, and so results that are true for them will generally extend to be true for any axes.

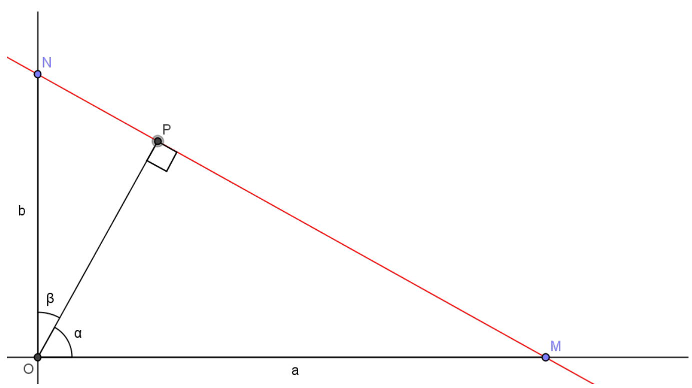
The reason we are so inclined to use rectangular axes over general ones is simplicity, once we introduce any concept of angles we must employ a number of additional cosine terms to account for the angle between the axes. However, the cosine of ninety degrees is zero so by working with rectangular axes we eliminate such terms. Consequently, going forward, we will treat all Cartesian systems we encounter in this thesis as rectangular.

The Abridged Notation for Lines

We now begin to introduce the abridged notation used by Salmon in his book. First we will build our system inside a Cartesian setting where, as the name suggests, our notation serves mainly as a shorthand. However, as we go on we will find that the system we build allows us to consider geometric ideas without the need for Cartesian axes, coordinates or measures. [Salmon Ch.4]

Lines by Perpendicular Distance

The concept of perpendicular distance is both simple and interesting, and it is what we shall work to place at the heart of a new method for writing lines. Geometrically it represents the shortest distance from a point to a line as the length of the segment joining the two. Furthermore, as the name suggests, it will always be perpendicular to the given line, forming a right angle.



We will now show how the perpendicular distance of an arbitrary line from the origin can be used to formulate an equation for the line. Let $OP = p$ be the length of the perpendicular and further let the angle $POM = \alpha$ denote the angle p makes with the x-axis.

As we have discussed before, all lines may be represented in the form

$$y = mx + b = -\frac{b}{a}x + b.$$

This line is no exception and we have also used the triangle formed by the axes and the line to obtain the gradient in terms of a and b. We can then rewrite it in the form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

We then multiply this equation by p to see

$$\frac{p}{a}x + \frac{p}{b}y = p.$$

Then, by inspection of the triangle and basic trigonometry, we have the equalities $\frac{p}{a} = \cos\alpha$ and $\frac{p}{b} = \cos\beta$ and so our equation for the line becomes

$$x\cos\alpha + y\cos\beta = p.$$

In rectangular coordinates, which we have agreed to work in, we can further simplify using the fact that $\beta = 90^\circ - \alpha$ which allows us to change our equation to

$$x\cos\alpha + y\sin\alpha = p.$$

For α between 0° and 360° we see that this allows us to consider all four quadrants of the plane and thus any possible line.

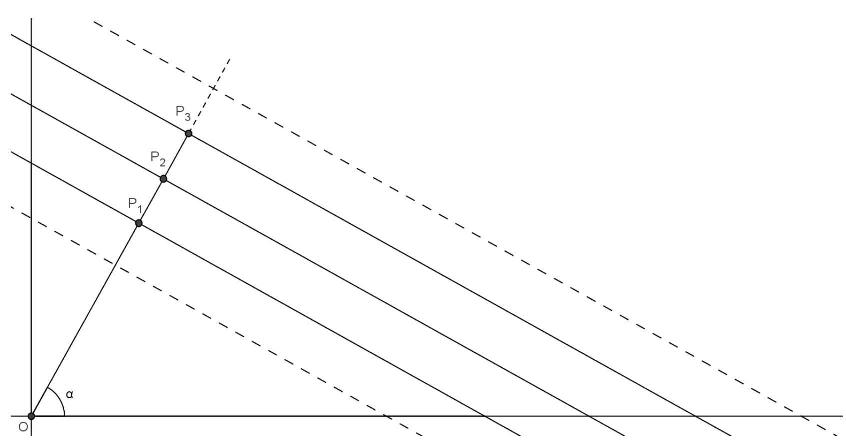
Abbreviation

We have now established that a line may be written in terms of its perpendicular distance from the origin called p and the angle this segment forms with the positive x axis called α.

$$\textit{Let} \quad x\cos\alpha + y\sin\alpha - p \quad \textit{be represented by} \quad \alpha.$$

Note how we choose to have it such that when our formula is equal to zero it represents the line; we discussed the utility of having the right hand side of the equation of line equal to zero when we first considered the equations of lines.

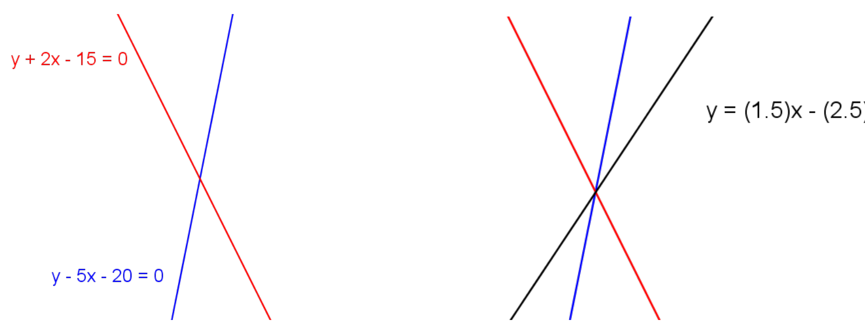
It is important to be aware that the angle α is not sufficient to define the line that we represent by α, we must understand that p is fixed even though it is not displayed. Without knowing p it is only possible to say that α represents one of infinitely many parallel lines whose perpendiculars form the correct angle with the x axis. This is demonstrated in the figure below, showing that for an angle α the values p_1, p_2 and p_3 all define a different line.



The first and most basic use of our new formulation is its utility in determining the equation of a line passing through the intersection of two distinct lines $\alpha = 0$ and $\beta = 0$, this will simply be, for some constant k,

$$\alpha - k\beta = 0.$$

To see that this represents a line we can write the terms out in full and consider the equation in a Cartesian setting. Let us consider an example to show this explicitly.



The left-hand figure shows two intersecting lines with their equations displayed, so we take $\alpha :\ y + 2x - 15$ and $\beta :\ y - 5x + 20$. We then consider

$$\alpha - k\beta = (y + 2x - 15) - k(y - 5x + 20)$$

$$= (1 - k)y + (2 + 5k)x + (-15 - 20k).$$

From this it is clear that $\alpha - k\beta = 0$ represents a line in the form $Ay + Bx + C = 0$ which may be be rewritten in the form $y = mx + c$ as

$$y = \left(\frac{2 + 5k}{k - 1}\right) x + \left(\frac{15 + 20k}{1 - k}\right).$$

In the right-hand figure k is taken to be equal to -1 and so the equation becomes $y = \frac{3}{2}x - \frac{5}{2}$ and this line is shown in black and passes through the intersection point of the two lines.

Now, to prove that this created line always passes through the intersection point, consider that regardless of the value of k if we take $\alpha = \beta = 0$ then the equation $\alpha - k\beta = 0$ is trivially satisfied. From this we know that there exists a point on the line lying on both $\alpha = 0$ and $\beta = 0$. The intersection of the lines $\alpha = 0$ and $\beta = 0$ is the only point in the plane lying on both those lines and so the line represented by $\alpha - k\beta = 0$ must pass through it.

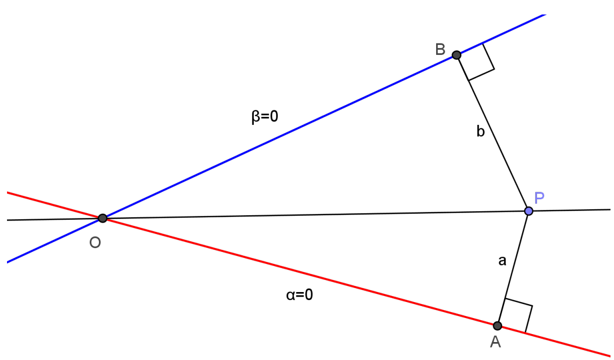
We now also adopt the notation that the point of intersection of the lines $\alpha = 0$ and $\beta = 0$ shall be called $\alpha\beta$. This will allow us to quickly and easily refer to the point of intersection of two lines which is a useful advantage of this notation.

A note of caution should be sounded here, we now have cause to sometimes view α as a line and sometimes as an angle, this ambiguity must be kept in mind as we proceed with more complex ideas. Similarly, we will later have cause to write $\alpha \cdot \beta$, a quadratic expression, as $\alpha\beta$ which we have here defined to be a point of intersection. We must ensure that we understand the meaning of these symbols and expressions wherever they are used as it may sometimes be a little less than obvious.

The Significance of k

When studying the geometric meaning of linear equations, or later quadratic equations, it is the constant terms that can be said to really decide the characteristics of the object. With this in mind we will now examine the meaning of the constant k that quietly appeared in the formulation of the previous section. In $\alpha - k\beta$ we understand that α and β are abbreviated names for the lines that are intersecting, so it is only k that remains to be examined.

Because of how we constructed the expression $\alpha - k\beta$ we know that the term α, shorthand for $x\cos\alpha + y\sin\alpha - p$, represents the perpendicular distance from the line represented by $\alpha = 0$ and similarly for the term β.



Then when we consider $\alpha - k\beta = 0$ we can surmise that k represents the constant ratio of the perpendiculars, labelled a and b for α and β respectively. Put another way, this k determines the relative position of our new line with respect to $\alpha = 0$ and $\beta = 0$. Our equation $\alpha - k\beta = 0$ represents a straight line through the intersection point $\alpha\beta$, labelled O, and

$$k = \frac{sinPOA}{sinPOB} = \frac{a}{b}$$

We also see that $\alpha + k\beta = 0$ will represent a line that *externally* divides angle into parts such that k still respects the same ratio.

One further point to note is that $\alpha - k\beta = 0$ is not normalized, meaning that multiplication by a constant will change the line represented here. This is because such a multiplication will alter the ratio k, though since α and β are normalized we will still obtain a line through the intersection $\alpha\beta$.

As we go on we will see that the abridged notation is often not concerned with normalization and so the fact that it is not always normalized will not impede us. However, for some things, such as the the computation of k, we must convert back to Cartesian, normalized coordinates.

Trilinear Systems

Our next step after considering the intersection of two lines is to consider a system of three. we shall show how any line in the plane may be written in terms of three known lines of reference, so long as those lines satisfy the necessary conditions. We will then look at how trilinear coordinates may be used to move us from abbreviation to the abridged notation proper, its uses and some of its more visible advantages. [Salmon 60]

We begin by considering three lines that form a triangle, in particular having no point common to all three, called α, β and γ. These will serve as our reference lines and we will show that any Cartesian line $ax + by + c = 0$ can be written in the trilinear form $l\alpha + m\beta + n\gamma = 0$.

We first write α, β and γ out in full to obtain

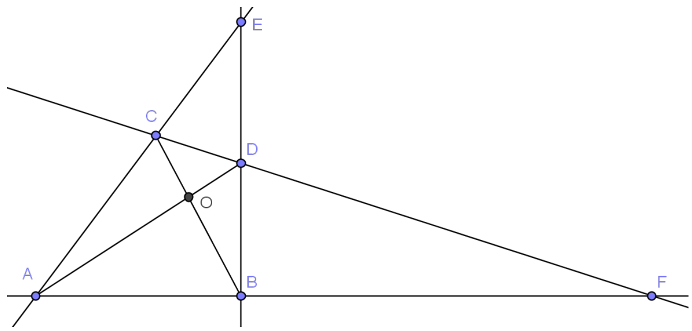
$$(l\cos\alpha + m\cos\beta + n\cos\gamma)x + (l\sin\alpha + m\sin\beta + n\sin\gamma)y - (lp + mp' + np'') = 0.$$

Then, so long as our three brackets are equal to the constants of our initial line, we will have the result. From here we can determine l, m and n such that

$$(l\cos\alpha + m\cos\beta + n\cos\gamma) = a,$$
$$(l\sin\alpha + m\sin\beta + n\sin\gamma) = b$$
$$(lp + mp' + np'') = -c.$$

This is because they form a system of three simultaneous equations in three unknowns and our requirement that they form a triangle ensures that these three conditions are not dependent upon one another, guaranteeing that a solution exists.

We will now consider an example system composed of six lines. We will show that every line in the figure can then be written in terms of $\alpha = 0$, $\beta = 0$ and $\gamma = 0$. In the figure the lines AD and BC are represented by segments as a way of keeping the figure less cluttered.



First we take the line AC as $\alpha = 0$, AB as $\beta = 0$ and BD as $\gamma = 0$. From the figure it is clear that these three lines form a triangle as we require. Then we may express AD as $l\alpha - m\beta = 0$ and BC as $m\beta - n\gamma = 0$ seeing that they are lines through the intersection of two of our reference lines. Lastly CD may be expressed as $l\alpha - m\beta + n\gamma = 0$, a line through the intersection of AD which we have seen to be $l\alpha - m\beta = 0$ and BD which is the reference line $\gamma = 0$. Thus we have expressed the equations of the other lines in terms of our chosen three. This is an important milestone for our ability to express a complex figure analytically.

Trilinear Coordinates

We have now seen that we can express the equation of any line in the form

$$l\alpha + m\beta + n\gamma = 0.$$

This tells us that any problem of lines may be solved by a set of equations expressed in terms of α, β and γ without any direct mention of x or y. This suggests that instead of a mere abbreviation for $x\cos\alpha + y\sin\alpha - p$ we may look at α as the property of the perpendicular distance from the line α.

We can imagine a system of *trilinear coordinates* in which the position of a point is defined by its distance from three fixed lines. In this system any straight line is defined by a homogeneous equation between these distances of the form

$$l\alpha + m\beta + n\gamma = 0$$

To motivate that this claim is true we will now compute an example using Cartesian coordinates. We begin by defining our three given lines in Cartesian terms.

$$\alpha = x\cos(30^\circ) + y\sin(30^\circ) - 4 = x\frac{\sqrt{3}}{2} + y\frac{1}{2} - 4,$$

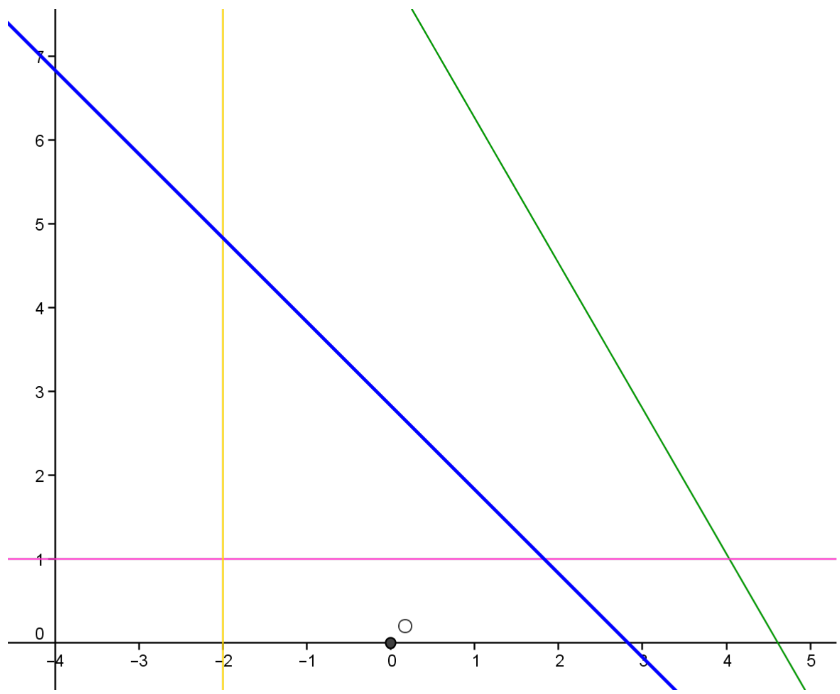
$$\beta = x\cos(90^\circ) + y\sin(90^\circ) - 1 = y - 1,$$

$$\gamma = x\cos(0^\circ) + y\sin(0^\circ) + 2 = x + 2.$$

These are shown in green, pink and yellow respectively on the figure below. Evidently these lines form a triangle so we may use them as the basis for a trilinear system within which we wish to demonstrate the equation of the blue line which will be called ϕ. It has Cartesian equation

$$\phi = x\cos(45^\circ) + y\sin(45^\circ) - 2 = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - 2.$$

From here we will show how to obtain a, b and c such that $\phi = a\alpha + b\beta + c\gamma$.



Note: *the y axis is placed off center to improve readability.*

Then we may formulate, by writing out α, β, γ and computing the trigonometric values, that in order to balance x terms, y terms and constant terms respectively we require the following three conditions.

$$(1): \qquad a\frac{\sqrt{3}}{2} + b(0) + c = \frac{1}{\sqrt{2}}$$

$$(2): \qquad a\frac{1}{2} + b + c(0) = \frac{1}{\sqrt{2}}$$

$$(3): \qquad -4a - b + 2c = -2$$

These are, as expected, three independent equations in three unknowns. Thus we can solve these simultaneously using standard algebraic methods.

$$(4) : (1) = (2) \qquad a\frac{\sqrt{3}}{2} + c = \frac{a}{2} + b \implies a(\sqrt{3} - 1) = b - c$$

$$(5) : (4) - (3) \qquad -4a - b + 2c + 2 = c + (c - b) - 4a + 2 = c - a(\sqrt{3} - 5) + 2 = 0$$

$$(6) : (5) - (1) \qquad (a\frac{\sqrt{3}}{2} + c - \frac{1}{\sqrt{2}}) - (c - a(\sqrt{3} - 5) + 2) = a(\frac{\sqrt{3}}{2} + \sqrt{3} - 5) - \frac{1}{\sqrt{2}} - 2 = 0$$

So we see, with approximate numerical values rounded to five significant figures, the values of a, b and c as we wanted.

$$a = \frac{4 + \sqrt{2}}{3\sqrt{3} - 10} \approx -1.1271$$

$$c = 2 + \frac{(4 + \sqrt{2})(\sqrt{3} - 5)}{3\sqrt{3} - 10} \approx 5.6832$$

$$b = \frac{(4 + \sqrt{2})(2\sqrt{3} - 6)}{3\sqrt{3} - 10} + 2 \approx 4.8581$$

We can see from this example that computing the exact numbers can be messy, but it is important to demonstrate that trilinear coordinates are reasonable to calculate. We have simply formulated the system rigorously to produce simultaneous equations and then solved them to obtain the values of the coefficients.

Initial Advantages of Trilinear

The first advantage of this trilinear coordinate system over the usual Cartesian coordinates is that we may choose three lines of a figure to be our fixed lines whereas in a Cartesian setting we may select only two axes. In addition, unless we choose to use oblique axes then in any system with no right angles we could only match one line to a Cartesian axis, but in trilinear coordinates we may select much more freely. These together allow us to express a system more compactly, as we demonstrated in the first example.

The focus of trilinear coordinates on perpendicular distance, while not an advantage in itself, provides a useful difference in perspective. Particularly, when considering the distance of one object from another, rather than from an object to the origin. This difference of perspective is an overarching attraction of trilinear and will allow us to clearly express geometric notions that would be difficult to express cleanly in a Cartesian setting.

It is also interesting to note that Cartesian coordinates may be considered as a particular case of trilinear, rather than an essentially different system. This can be most easily seen by showing that Cartesian equations are homogeneous even if they are not written as such. When we write $x = 3$ as a Cartesian equation it means that x is equal to three meters or three inches, in general it is three times some linear unit that we will call z. This linear unit is the hidden component that allows us to recognize that the Cartesian equation of a line may be written as $Ax + By + Cz = 0$.

Further, consider that when a line is infinitely far from the origin it must take the form $z = 0$ and so we understand that Cartesian coordinates are trilinear. We take the coordinate axes as our first two lines of reference and a third line at an infinite distance. It is important to reinforce here that we are not talking about a different plane or different objects, merely a different way of labelling objects and talking about them.

The Line Through Two Points

As a demonstration that some geometric constructs are not strictly easier when considered in a trilinear setting we will now look at a specific property in Cartesian coordinates, the line through two points. In Cartesian coordinates we may construct an equation $y = mx+c$ given two points with coordinates (x_1, y_1) and (x_2, y_2) using the formula $m = \frac{y_2-y_1}{x_2-x_1}$ and then substituting in values to solve for c.

We will now examine how this process translates to our new notation, finding the line in the form $l\alpha+m\beta+m\gamma = 0$. Let us again take two arbitrary Cartesian points and call them (x_1, y_1) and (x_2, y_2). Our first step will be to take α_1 to be $x_1 \cos\alpha + y_1 \sin\alpha - p = 0$, that is the equation of α when $x = x_1$ and $y = y_1$, and then take β_1 and γ_1 defined similarly. The key here is to note that (x_1, y_1) will satisfy $l\alpha + m\beta + n\gamma = 0$ if and only if we have

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0.$$

This can be seen easily in each independent part, for example with $x = x_1$ and $y = y_1$we have the equivalence

$$l\alpha_1 = l(x_1 \cos\alpha + y_1 \sin\alpha - p) \iff l(x \cos\alpha + y \sin\alpha - p) = 0.$$

We may also, by the same reasoning, take

$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0$$

to represent the condition that the second point lies on our line. Here α_2, β_2 and γ_2 are all defined in the same fashion as we defined α_1, β_1 and γ_1.

Solving for $\frac{l}{n}$ and $\frac{m}{n}$, and substituting back into our given $l\alpha+m\beta+n\gamma = 0$ form we obtain

$$\alpha(\beta_1\gamma_2 - \gamma_1\beta_2) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0.$$

Since equations in trilinear coordinates are homogeneous we are not concerned what the exact values are here, only with their mutual ratios. Thus we may, if we choose, write $\rho\alpha_1$, $\rho\beta_1$ and $\rho\gamma_1$ instead of $\alpha_1, \beta_1, \gamma_1$ and observe that the common coefficient vanishes when we divide one by another.

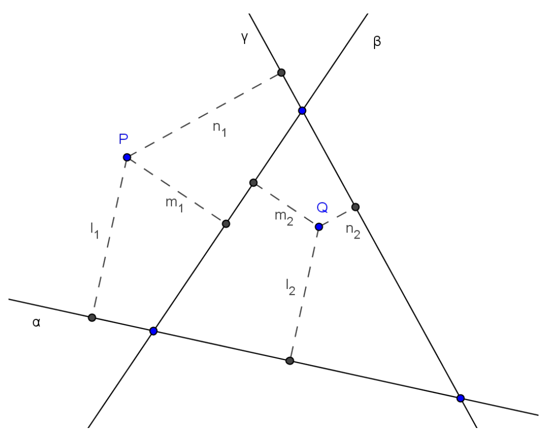
The Trilinear coordinates of a point

One other area where Cartesian coordinates are very effective is in the precise description of points; the whole system is build on points where trilinear is focused on objects such as line and conics. However, a point may be described relative to three reference lines α, β, γ by $(l : m : n)$, three numbers representing the perpendicular distance of the point from each reference line. These numbers are exactly the coefficients of the terms of the equation

$$l\alpha + m\beta + n\gamma = 0.$$

We also neatly avoid the issue we observed with Cartesian in that for distance to be meaningful it must have units of reference. We do this by only considering the ratio of the point between two lines.

For example, the point p will lie between α and β at a ratio of l to m and since we are only concerned with ratios any common measure between distances will cancel.



The figure shows one point inside the triangle and one outside it, in any case the same principle applies. We can see the point p may be described as lying at $(l_1 : m_1 : n_1)$, meaning that it lies in, for example, ratio $\frac{l_1}{m_1}$ between α and β.

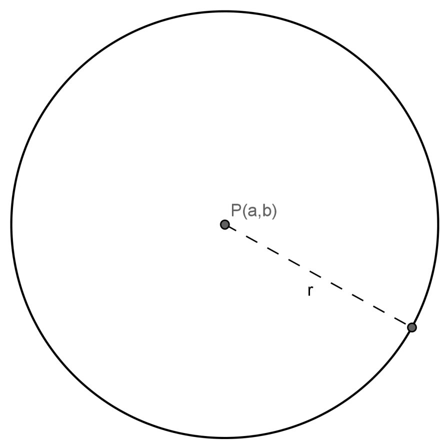
This method of representing points is also widely used in the field of projective geometry with the name *homogeneous coordinates*. Projective geometry is an area of mathematics that this thesis touches on in passing, but which we do not have the time to pursue further.

We will not often have cause to describe arbitrary points in this thesis, instead we are more concerned with points of intersection or other distinguished points. We can and will identify such points by those distinguishing characteristics instead of this method of ratios, as this will prove both easier and more insightful. That said, it is important to be clear that even with a focus on lines the trilinear system is perfectly capable of dealing with points in general.

Circles

So far we have seen both how we treat some basic linear geometry of the plane in the usual Cartesian way and how we can view things differently using the beginnings of Salmon's abridged notation. We have looked at points, lines and triangles and now we move on to the next object in order of complexity. The circle, a particular case of conic sections, that we use as a stepping stone to a more general discussion of conics. [Salmon Ch.6]

We will find that, similar to computing k, distinguishing a circle from a general conic form requires us to invoke Cartesian coordinates. This will mean that once we begin investigating circles in the abridged notation we will almost automatically transition to talking about general conic sections.



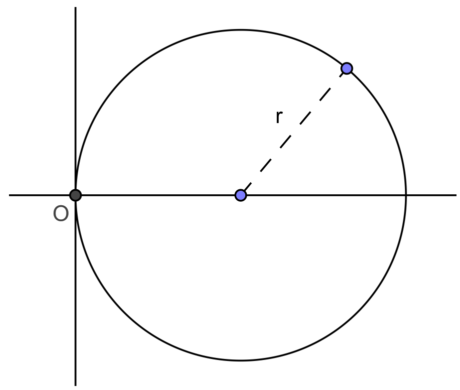
When we represent a circle in Cartesian coordinates we do so using two quantities, it's center point and it's radius. We see that this will require three numbers, the two coordinates of the center and the distance from the center to the edge. We arrange them in the second degree equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

Hence, for any point with coordinates (a, b) we have constructed the circle with radius r around that point. There are also two cases of special interest when speaking of Cartesian circles that we will cover here. First, a circle centered at the origin $(0, 0)$ may be written simply

$$x^2 + y^2 = r^2.$$

Alternatively, if the origin is placed at the leftmost point of the circle then the x axis passes through the center and the y axis touches the circle tangentially at the origin.



In this configuration $a = r$ and $b = 0$, so our equation is $(x - r)^2 + y^2 = r^2$. When we expand the bracket we obtain an r^2 term and so we may cancel r^2 from both sides leaving us, after a small rearrangement, the form

$$x^2 + y^2 = 2xr.$$

These forms are useful because they produce simplified algebraic expressions which are easier to work with. They are ways to simplify the notation for circles, similar to how using rectangular axes simplifies most expressions. As with lines, we are not interested in the general study of circles in this thesis, but we seek to gain an appreciation for how they are described.

Difference from a General Conic

We have mentioned that circles are a particular case of conic sections, or conics, and while we leave more detailed discussion of conics for their own section we will discuss here the ways that circles differ from other kinds of conic.

Fact. *All conics are second degree curves, this means that they are represented by a Cartesian equation of the general form*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The name "second degree" refers to the power of two in the form above. We see in the equation all the first degree terms; x, y and the constant term c. To these are added the quadratic, second degree terms x^2, y^2 and xy to give us a fully general second degree equation. We see that where a first degree equation had three coefficients this second degree equation has six.

To be a circle however, we require the conditions that

$$h = 0 \quad \textit{and} \quad a = b.$$

Observe the expansions below, beginning with the equation of a circle as we have discussed it.

$$(x-a')^2+(y-b')^2-r^2 = x^2-2a'x+a'^2+y^2-2b'y+b'^2-r^2$$
$$= x^2+y^2+2(-a')x+2(-b')y+(a'^2+b'^2-r^2)=0.$$

Then, multiplying this normalized Cartesian equation by a constant and renaming the constant terms we obtain the expected formula

$$ax^2+ay^2+2gx+2fy+c=0.$$

These conditions have led to us only needing four coefficients. Indeed if we consider a normalized form we may divide by a and be left with only three coefficients; x, y and the constant term.

Abridged Notation for a Circle

We now move to look at how we may express a circle in a Trilinear coordinate setting. From the outset we must realize that we have no concept of radius in this system and so from the very beginning we must approach the construction of a circle from a number of different perspectives. These perspectives will have in common the use of an *inscribed* or *circumscribed* shape to define the circle. [Salmon Ch.9]

An inscribed shape is contained inside another shape and touching each side once. A circumscribed shape passes outside another, fully containing it and touching each vertex once.

The first perspective we shall consider is the circle as a conic circumscribing a quadrilateral.

Proposition. *Given a quadrilateral defined by the four lines that form its edges α, β, γ and δ. Then the equation of a circumscribing conic is of the form*

$$\alpha\gamma = k\beta\delta.$$

Furthermore, if the quadrilateral satisfies certain necessary conditions then this conic will precisely be a circle.

Here k is a constant. Since each side of the equation has two linear terms multiplied together we will obtain second degree terms, ensuring we do obtain a conic. Additionally, the equation is constructed in such a way that for any of the four following conditions it will be trivially satisfied.

$$\alpha = 0 \text{ and } \beta = 0 \qquad \alpha = 0 \text{ and } \delta = 0$$

$$\beta = 0 \text{ and } \gamma = 0 \qquad \gamma = 0 \text{ and } \delta = 0$$

These represent the intersection points of pairs lines, the points $\alpha\beta$, $\alpha\delta$, $\beta\gamma$ and $\gamma\delta$ respectively. These are precisely the vertices of our quadrilateral. This means that our formulation gives us a conic that passes through the four corners of our quadrilateral.

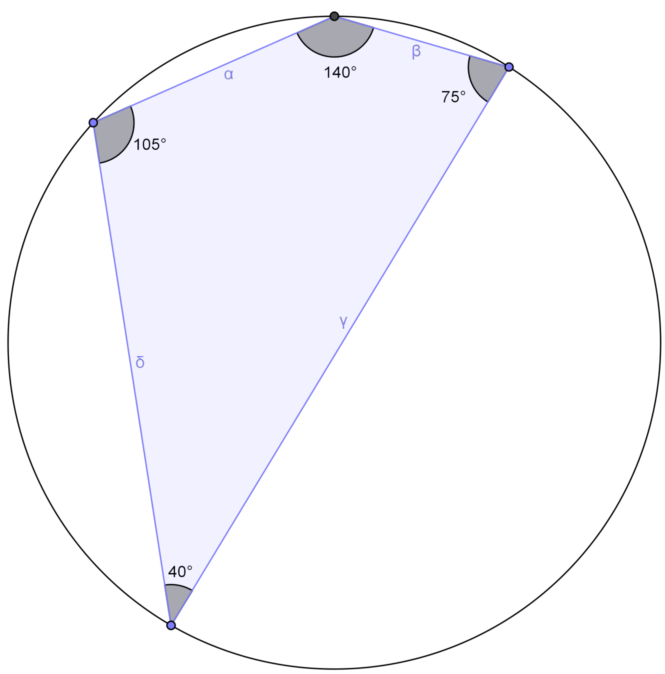
From here we must impose the condition that our conic is exactly a circle. We do this as we did in the last section, beginning by writing our equation $\alpha\gamma = k\beta\delta$ at full length

$$(x\cos\alpha + y\sin\alpha - p)(x\cos\gamma + y\sin\gamma - p'')$$
$$= k\,(x\cos\beta + y\sin\beta - p')(x\cos\delta + y\sin\delta - p''').$$

We can then multiply out the brackets to obtain the complete second degree equation. We now use the conditions we found in the last section, equating the coefficients of x^2 and y^2 and setting the coefficient of $xy = 0$. This gives us the conditions

$$\cos(\alpha+\gamma) = k\cos(\beta+\delta) \quad \textit{and} \quad \sin(\alpha+\gamma) = k\sin(\beta+\delta).$$

We then square these equations and add them together to eliminate the trigonometric terms using the identity $\cos^2\theta + \sin^2\theta \equiv 1$. We find that in doing this we require $k = \pm 1$ and $\alpha+\gamma = \beta+\delta$ or $\alpha+\gamma = 180^\circ + \beta + \delta$.



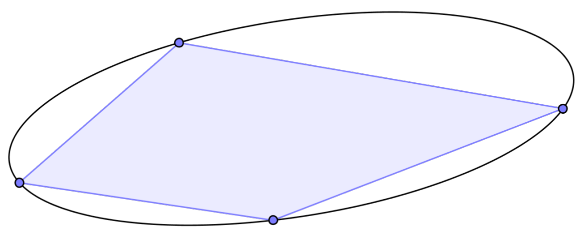
The figure shows an example with the angles of the quadrilateral displayed and we can see that the opposite angles both sum up to the same amount.

It should be noted that the internal angles of our quadrilateral are not the angles α, β, γ and δ as those are the angles between the perpendicular distance of their respective lines to the origin and the x axis. However, due to the relation of those angles to the angles of the quadrilateral, the property $\alpha + \gamma = \beta + \delta$ is reflected here. It can also be seen that if the origin lies inside our quadrilateral then we require $k = -1$ and if it lies outside then we require $k = +1$.

Thus, we have shown that we may construct a circle around a quadrilateral so long as the quadrilateral satisfies certain conditions and that the circle will have the equation

$$\alpha\gamma = k\beta\delta.$$

Further, for a quadrilateral not satisfying those conditions the construction will not produce a circle, instead obtaining some manner of general conic, most usually an *ellipse*. This means our construction intuitively and immediately extends to the construction of the circumscribing conic of a quadrilateral. Hence we have proved all the claims of the proposition.



The figure above shows an example of the circumscribing conic of an arbitrary quadrilateral. In this particular case we are shown an ellipse, although based on other k values in $\alpha\gamma = k\beta\delta$ we could obtain other conic forms.

Conic Sections

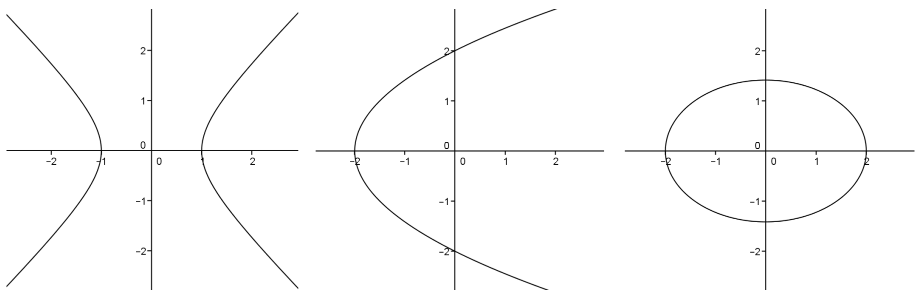
An important idea to take forward from the last section is that our abridged notation does not easily distinguish between types of conic section. We need to revert to Cartesian coordinates in order to make the distinction that a general conic is particularly a circle. [Salmon Ch.10]

We have now used the terms conic and conic section several times and so we will discuss in this section exactly what conics are before we move on to show other ways of constructing them in the abridged notation. The first classical understandings of conic sections are known to date back to ancient Greece. There are many interesting results concerning conics that we will not have room to discuss in this thesis. We shall instead content ourselves with a definition and brief history, neglecting to examine the properties of these objects.

Definition. *A conic section is a plane curve of the second degree. Thus it has a Cartesian equation in the general form*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Traditionally the three main types of conic section are the *hyperbola,* the *parabola* and the *ellipse.* An example of each is shown below and at least the latter two may well be somewhat familiar to readers.



The equations of the curves shown in the figure are

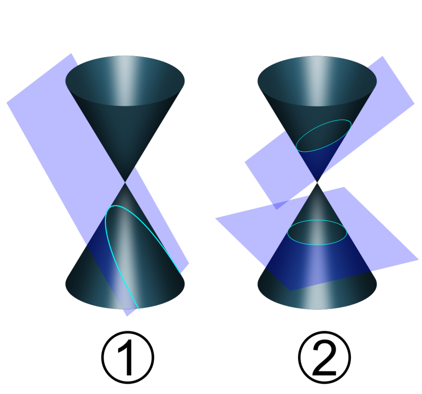
$$Hyperbola: \quad x^2 - y^2 = 1$$

$$Parabola: \quad y^2 - 2x = 4$$

$$Ellipse: \quad x^2 + 2y^2 = 4$$

The circle is sometimes considered as a separate fourth type of conic due to its importance in other areas of study, however we will consider it simply as a special case of an ellipse.

The origin of the name "conic section", often shortened to "conic", arises from a double cone construction. When cutting a double cone with a plane, we may obtain these three distinct figures in the plane along with their special cases. This is demonstrated in the figure below for the main cases (1) A parabola, (2) An ellipse, upper, and a circle, lower, and (3) A hyperbola.



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Three other cases are of note; first, a plane passing through the center point where the two cones touch and cutting the cones top to bottom will yield a figure of two straight lines intersecting at a point. The two lines may coincide giving the appearance of a single line if the plane runs tangent to the cone. Alternatively we obtain a single point if the plane is somewhat horizontal, touching the cones only at their center point.

These figures are special cases of the usual conics, similar to the circle. The cases of two lines, coincident and not, will be used extensively as we further develop our abridged notation.

The study of conics is ancient and contains a vast wealth of interesting topics, however we do not need these details for our purposes. We return now to methods of representing conics in the abridged notation.

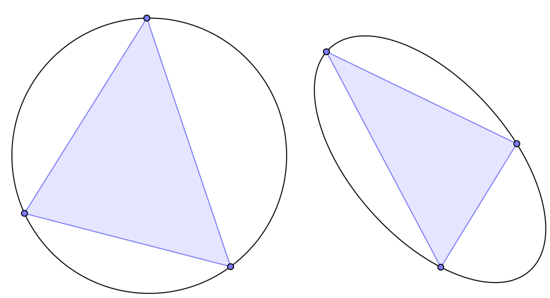
Conics Defined by Triangle

In a previous section we saw how the circumscribed conic of a quadrilateral may be constructed, and that under certain conditions the conic may be a circle. For the next construction we move from a quadrilateral to a triangle, this relates much more closely to the trilinear system we have been building.

Proposition 1. *Given three lines $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ forming a triangle the circumscribed conic of the triangle $\alpha\beta\gamma$ has the form*

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

Furthermore, under certain necessary conditions this conic will be a circle.



This equation, similar to the quadrilateral case, contains products of two linear equations which result in second degree terms meaning that some conic is represented here. Further, it is once again constructed so that the corners, or *vertices*, of the triangle will trivially satisfy the equation. This means that the conic must pass through those vertices. Hence we have the circumscribing conic, as claimed.

Next, to find when this equation represents a circle, we equate the coefficients of x^2 and y^2 and set the coefficient of xy to be zero. From this we obtain the following conditions.

$$l\cos(\beta+\gamma)+m\cos(\gamma+\alpha)+n\cos(\alpha+\beta)=0,$$
$$l\sin(\beta+\gamma)+m\sin(\gamma+\alpha)+n\sin(\alpha+\beta)=0.$$

We will use a lemma to finish the proof of this proposition.

Lemma. *Given two lines $l\alpha'+m\beta'+n\gamma'=0$ and $l\alpha''+m\beta''+n\gamma''=0$, the following proportionalities hold for the coefficients l, m and n.*

$$l\propto\beta'\gamma''-\beta''\gamma'\Longrightarrow l\propto\sin(\beta-\gamma)$$
$$m\propto\gamma'\alpha''-\gamma''\alpha'\Longrightarrow m\propto\sin(\gamma-\alpha)$$
$$n\propto\alpha'\beta''-\alpha''\beta'\Longrightarrow n\propto\sin(\alpha-\beta)$$

This result follows from the use of the angle sum identities from trigonometry which we state below.

$$\sin(\theta\pm\lambda)\equiv\sin\theta\cos\lambda\pm\sin\lambda\cos\theta$$
$$\cos(\theta\pm\lambda)\equiv\cos\theta\cos\lambda\mp\sin\theta\sin\lambda$$

First we must combine our two conditions into a single long equation, using the identities, which we see below.

$$l(\cos(\beta+\gamma)-\sin(\beta+\gamma))+m(\cos(\gamma+\alpha)-\sin(\gamma+\alpha))+n(\cos(\alpha+\beta)-\sin(\alpha+\beta))=0$$

The identities then allow us to reduce the result to the more manageable form

$$\beta\gamma\sin(\beta-\gamma)+\gamma\alpha\sin(\gamma-\alpha)+\alpha\beta\sin(\alpha-\beta)=0.$$

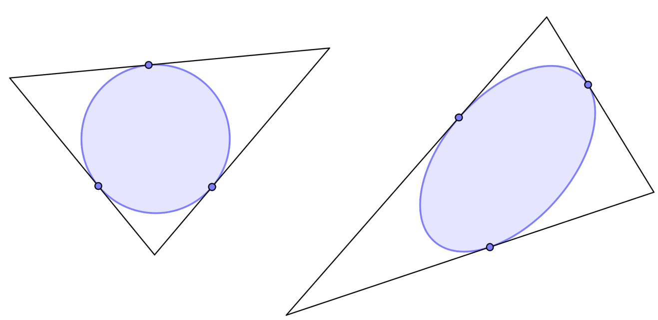
Thus we have found the conditions for the circumscribed conic to be a circle and have proved the first proposition of this section.

We shall next investigate the equation of an inscribed conic of a triangle.

Proposition 2. *Given three lines$\alpha=0$, $\beta=0$ and $\gamma=0$ forming a triangle the equation for an inscribed conic of the triangle $\alpha\beta\gamma$ is*

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0.$$

Furthermore, under certain necessary conditions this conic will be a circle.



The sides of the triangle are tangents to the conic, meaning they meet the curve in two coincident points precisely as we need them to. This is because when we set $\gamma = 0$ in the equation we obtain the perfect square

$$l^2\alpha^2 + m^2\beta^2 - 2lm\alpha\beta = (l\alpha)^2 + (m\beta)^2 = 0.$$

We may then also obtain perfect squares for the other sides in the same manner. Hence we have the equation of an inscribed conic and we move on now to use Cartesian consideration to discover when this conic is a circle.

We do this in the same way as before. The condition that the equation represents a circle may be written at full length as

$$m^2\sin^2 C + n^2\sin^2 B + 2mn\sin B\sin C$$

$$= n^2\sin^2 A + l^2\sin^2 C + 2nl\sin A\sin C$$

$$= l^2 sin^2 B + m^2 sin^2 A + 2lm\sin A\sin B.$$

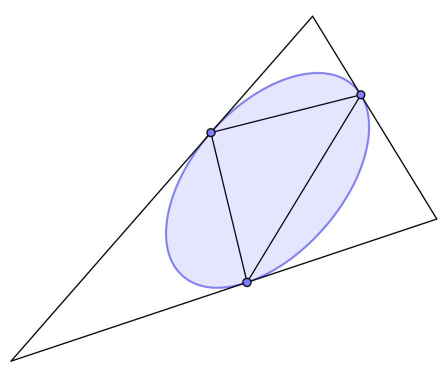
Or more compactly, by taking square roots, in the form

$$m\sin C + n\sin B = \pm(n\sin A + l\sin C) = \pm(l\sin B + m\sin A).$$

It is then clear that there are four possible circles that may be described as touching the sides of a given triangle, obtained by the varying signs in the short form of the equation. In the case of choosing both signs to be positive the equation will be that of the inscribed circle;

$$\cos\frac{1}{2}A\sqrt{\alpha} + \cos\frac{1}{2}B\sqrt{\beta} + \cos\frac{1}{2}C\sqrt{\gamma} = 0.$$

This result may also be seen as following from the previous result. Consider the figure below



From this it is simple to see that the inscribed conic of the outer triangle is the circumscribed circle of the inner triangle. We will not deal with this construction algebraically as we have already treated both parts separately.

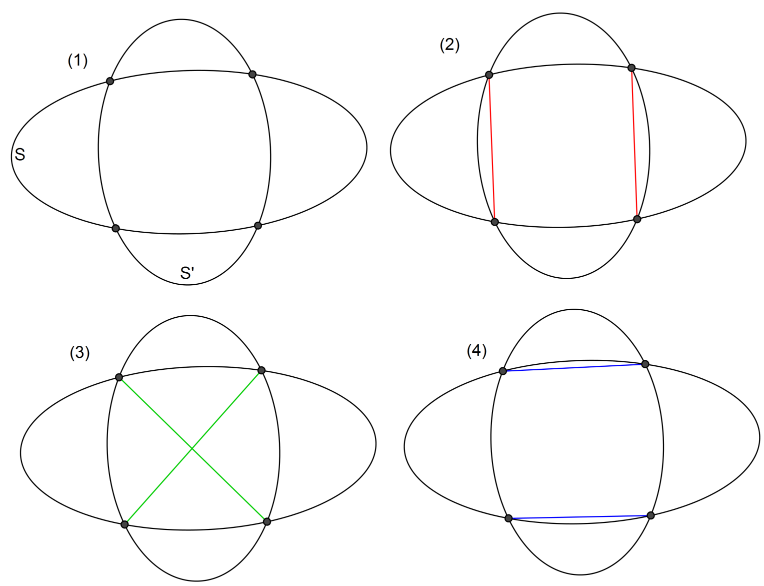
Conics in Trilinear Coordinates

We have now seen how we may formulate the circumscribed conic of a quadrilateral or triangle, and the inscribed conic of a triangle. Next we will consider the intersection of two conics. Take $S = 0$ and $S' = 0$ to be two conics, then the equation of any conic passing their four, real or imaginary, points of intersection can be expressed in the form $S - kS' = 0$. [Salmon Ch. 15]

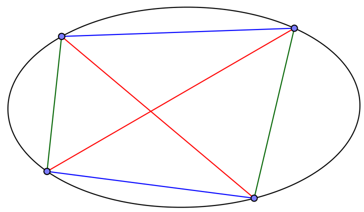
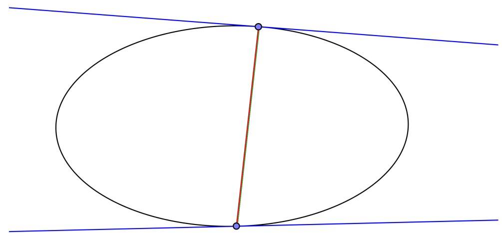
In the case of our abridged notation for a line; two lines will always intersect in one point, parallel lines meeting at infinity, and we may express any third line through that point as a linear combination of the given two. Two conics always, in a similar manner, intersect in four points and we can express any other conic through those four points as a linear combination of the given two.

For each value of k there will be a fifth point satisfying the resulting equation $S - kS' = 0$ and so we conclude that an equation of this form represents the conic determined by five points. This again refers back to our treatment of lines and we now investigate the significance of the constant k.

In particular we are interested in the values of k for which $S - kS' = 0$ represents a pair of straight lines, rather than any other conic form. There are three possible configurations displayed in the figure below beside an unembellished figure of S and S'.



We can clearly see the possible cases, even without knowing the corresponding k values. From here we consider the case where $S - k\alpha\beta = 0$, this represents a general conic intersected in four points by two lines with the linear combination of these two giving another conic.

If we then suppose that the lines α and β coincide we will have only two intersection points and our equation becomes $S - k\alpha^2 = 0$. In this situation we have a conic that has double contact with S and α is the chord of intersection. Further, we observe that we could formulate the same conic as $S - kTT' = 0$ where T and $T'$are the two tangents at the points of intersection, shown in blue on the figure. The purpose of showing the two above figures side by side is to underline the way in which the six diagonals become a single chord and two tangents in this special case. To state it clearly the result here is

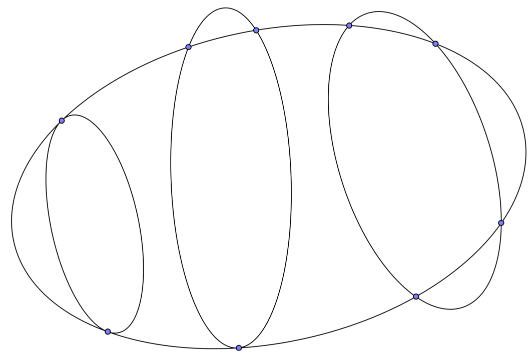
$$S - k_1\alpha^2 = S - k_2TT' = 0.$$

Knowledge of the chord of intersection and these particular tangents will be useful in the construction and analysis of more complex systems as we move forward. The simple representation of these geometrically significant quantities is a key strength of the abridged notation and one on which we will rely when we begin to prove more complex results. At this point we have assembled most of the understanding necessary to apply our abridged notation to the four results that we have been working towards.

Imaginary and Infinite Intersections

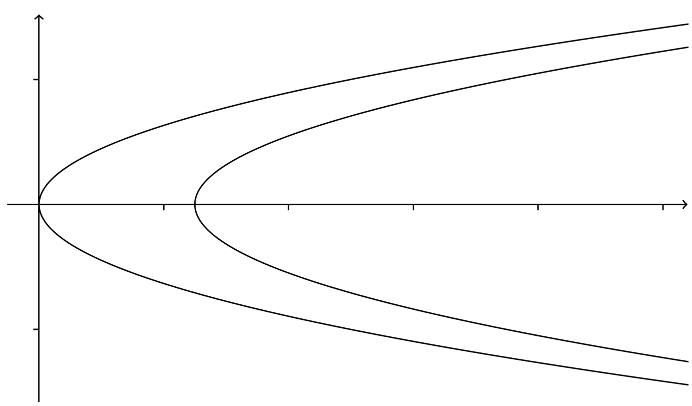
So far in this thesis we have referred several times to the idea that conics and lines always intersect in a fixed number of points. In cases where these points are not obvious we have used, without proper explanation, a number of reasonings to extend our claim to these cases. We will now examine in detail these reasonings so that we are clear on how they justify our claims. Particularly, we will consider cases of conic sections as they concern us most keenly as we go forward.

The first way in which we may encounter seemingly fewer intersection points than expected is that two or more points, which are in general distinct, coincide in a particular case. In such a situation we have only to note that there are two points, they are simply overlapping each other. This is shown below for conics, specifically here ellipses, appearing to have two or three points of intersection.



The second case is less intuitive, a point which does not appear to lie anywhere in the real plane may lie at infinity. The most common case where we see this is the assertion that two parallel lines have a point of intersection at infinity.

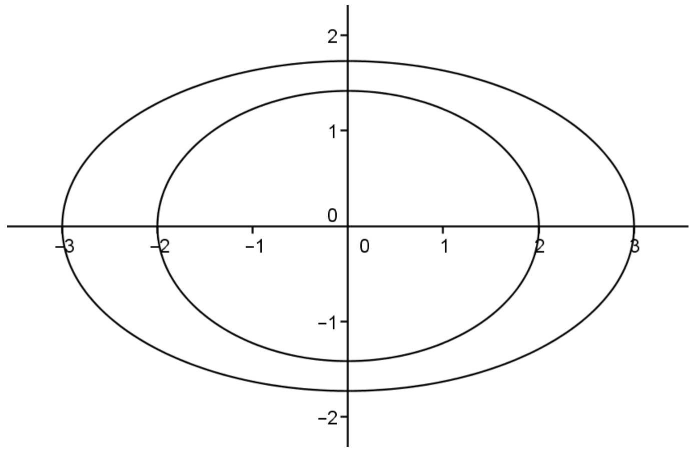
This can similarly apply to parabolas and hyperbolas which also extend to infinity in the same manner as lines, in the figure below we see the parabolas $y^2 = 4x$ and $y^2 = 4x - 50$.



As they grow the distance between the two shrinks but they will never touch for any finite value. In such a case it is natural to suppose that they instead meet at some point at infinity. In the case of these parabolas, there will be two distinct intersections at infinity each representing two coincident points, showing that the cases we are examining here may overlap.

The third and final case concerns two non infinite shapes with no apparent intersection points. Usually in this thesis this means ellipses as they are the most normal finite conic form. In this situation we claim that the points of intersection are imaginary. This notion may seem unlike mathematics to readers who have not encountered the concept previously, but it is based on the precise use of the theory of imaginary numbers that begins with the statement $\sqrt{-1} = i$, where i is the imaginary unit.

To demonstrate the rigour of this idea we consider an example using two ellipses, $x^2 + 2y^2 = 4$ and $x^2 + 3y^2 = 9$ shown in the figure below, having no real intersection points.



We now solve algebraically, beginning by equating our two ellipses

$$x^2 + 2y^2 - 4 = x^2 + 3y^2 - 9.$$

We then cancel the x^2 terms and solve for y, finding that $y^2 = 5$ and so $y = \pm\sqrt{5}$. Then by substituting back into either ellipse, $x^2 + 2(5) = 4$ or $x^2 + 3(5) = 9$, we find that $x^2 = -6$. We then compute the root of this negative number as follows.

$$x = \pm\sqrt{-6} = \pm i\sqrt{6} \approx \pm 2.45i$$

The approximate decimal value being rounded to three significant figures. Thus we have the four imaginary points of intersection

$$(i\sqrt{6},\sqrt{5}),\ (i\sqrt{6},-\sqrt{5}),\ (-i\sqrt{6},\sqrt{5})\ \mathit{and}\ (-i\sqrt{6},-\sqrt{5}).$$

This may be neatly shortened to $(\pm i\sqrt{6}, \pm\sqrt{5})$ since our intersection points happen to form a rectangle with its center at the origin. This happened because the example chose the simplest of ellipses to work with. In general imaginary points of intersection are just as disparate as their real counterparts.

We have now, in detail, considered the ways in which our results concerning intersections of conics and lines may be extended using the concepts of coincident points, points at infinity and imaginary points. These notions allow us to speak in full generality about the four intersection points of distinct conics.

Relationships Between Conics

We will now construct a useful lemma on a system of conics. Recall that given two intersecting conics we will have four intersection points and the chords of intersection are the six lines joining pairs of those points. [Salmon 264]

Proposition. *Given two conics, each having double contact with a third conic, all the chords of intersection between pairs of these three will pass through a single point.*

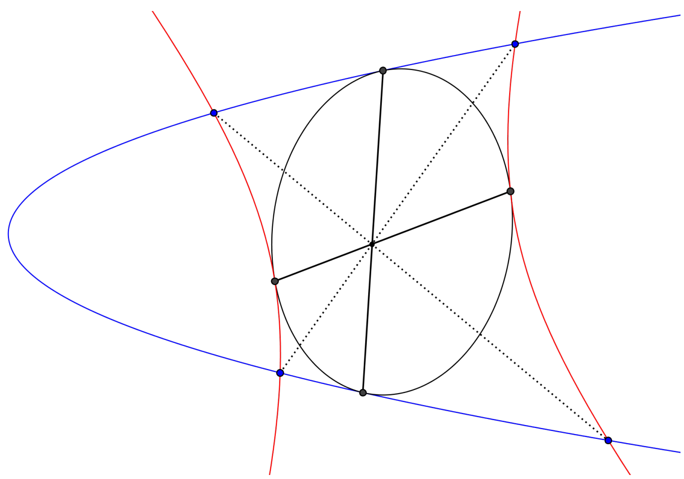
We begin by defining our conics to be

$$S = 0, \qquad S + L^2 = 0, \qquad S + M^2 = 0.$$

By definition the conics $S + L^2 = 0$ and $S + M^2 = 0$, each have double contact with the third, $S = 0$. Recall that given this double contact things reduce from six chords in general to one chord and two tangents. The chords in this case are L and M, connecting the two intersection points of each contained conic with $S = 0$. From here we observe that

$$S' - kS'' = (S + L^2) - (S + M^2) = L^2 - M^2 = 0.$$

This gives us the pair of lines $L \pm M = 0$ as the two chords of intersection of $S + L^2 = 0$ and $S + M^2 = 0$. These chords must both pass through the points of intersection of L and M since they have the form $\alpha - k\beta = 0$. Thus we have proved the proposition.



In the figure above we see that $S = 0$ is an ellipse sharing two double contacts with both the hyperbola $S + L^2 = 0$ and the parabola $S + M^2 = 0$. The conics are coloured black, red and blue respectively with $L = 0$ and $M = 0$ shown as solid lines and $L \pm M = 0$ shown as dotted lines.

Building on this case of three conics we will establish an extension of the result to a system of four conics. This lemma follows quickly from the proposition and will prepare us for the the first classical theorem.

Lemma. *Consider three conics each having double contact with a fourth, their chords of intersection will pass three by three through the same points. These chords will thus form the sides and diagonals of a quadrilateral.*

We begin by naming the four conics. Let

$$S = 0, \quad S + L^2 = 0, \quad S + M^2 = 0, \quad S + N^2 = 0.$$

By use of our previous result we find that the chords, in their groups of three, will be

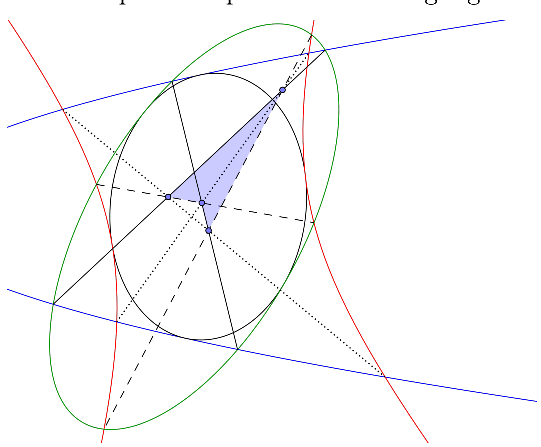
$$L - M = 0, \quad M - N = 0, \quad N - L = 0;$$
$$L + M = 0, \quad M + N = 0, \quad N - L = 0;$$
$$L + M = 0, \quad M - N = 0, \quad N + L = 0;$$
$$L - M = 0, \quad M + N = 0, \quad N + L = 0.$$

We may be certain that these groups of three lines all intersect by simply considering that they are linearly dependent. That is, that we may express one as a linear combination of the other two, as $\alpha - k\beta$. In each of these cases we may easily observe that this is true, for example

$$(L + M) - (M + N) = N - L = 0.$$

Further, the lines represented here present every possible connecting line through pairs of these four points so it is certain that we may select lines from them that form a quadrilateral and that the remaining lines will be the diagonals of this quadrilateral. Thus the lemma is proved.

The figure below shows our example from the case of three conics with the addition of a second ellipse, $S + N^2 = 0$, shown in green. The four common points are shown and one possible quadrilateral is highlighted.



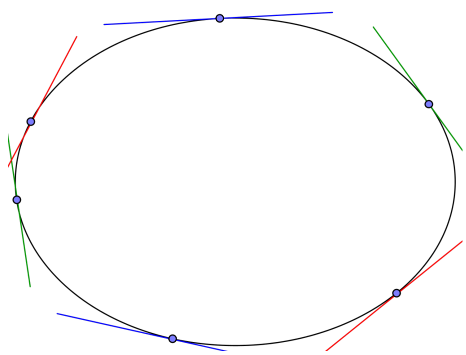
As it stands the applications of this result are difficult to appraise because of the generality of the statement, but it has many particular cases depending on which, if any, of the conics we suppose to factor into lines.

Brianchon's Theorem

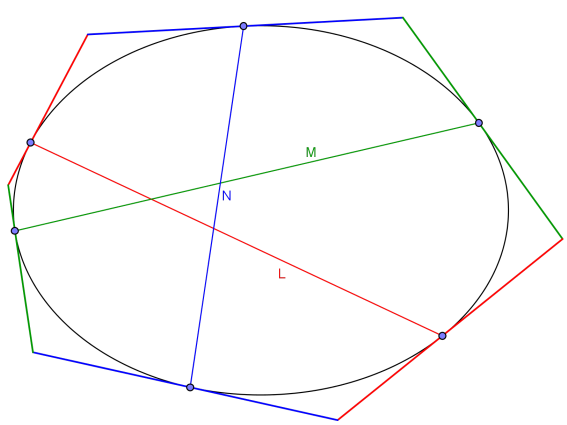
We are now ready to consider the four results that we have been leading up to. These results will demonstrate the utility of the abridged notation and the trilinear coordinates that we have developed throughout this thesis. We begin with Brianchon's theorem, named after the French mathematician Charles Julien Brianchon, who proved it in 1810. [Salmon 265]

Theorem. *For every hexagon circumscribing a conic the principal diagonals, those connecting opposite vertices, intersect in a point.*

We address this as a specific case of our lemma from the previous section. Begin by considering the figure below showing an ellipse which we call $S = 0$ and three pairs of lines with each line tangent to the ellipse. In this way each of these three pairs of lines is a conics having two double contact points with the ellipse and so we may express them in the form $S - kTT' = 0$, where T and T' are tangents to $S = 0$.



From this we have obtained three pairs of tangents that when taken together form the six sides of a hexagon circumscribing the ellipse. We also note the chords of intersection of each pair of lines with the ellipse, naming them L, M and N.



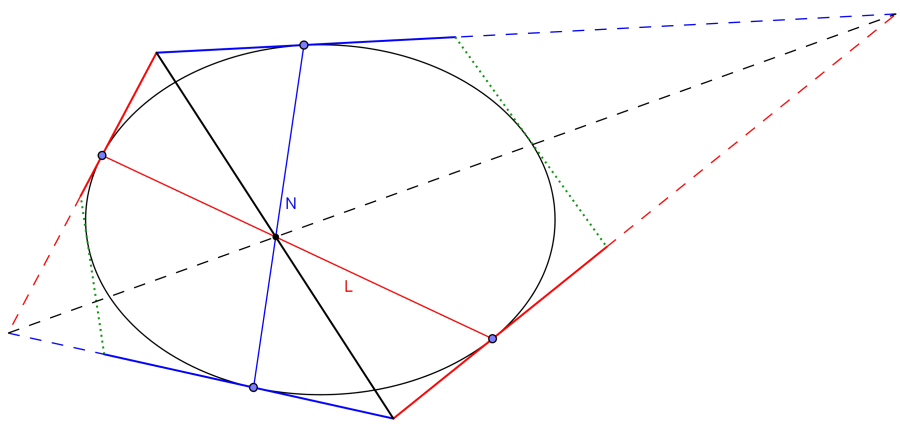
Recall now the key result of a previous section that

$$S - kTT' = S - \alpha^2 = 0.$$

This means that we may represent the pairs of tangent lines in terms of their chords of intersection which we have just labelled L, M and N. Thus we have our four conics in exactly the form needed for us to apply the lemma from the previous section.

$$S = 0, \quad S + L^2 = 0, \quad S + M^2 = 0, \quad S + N^2 = 0$$

We have now also found the hexagon circumscribing the ellipse and so it makes sense to talk about the diagonals of this hexagon. To see that the principal diagonals meet in a point, we now apply our lemma to this set-up of four conics. It will become clear that one of the four sets of three lines are exactly the principal diagonals of our hexagon. To see this we neglect $S + M^2 = 0$ for a moment and look at the three remaining conics using the proposition that underpins the lemma.

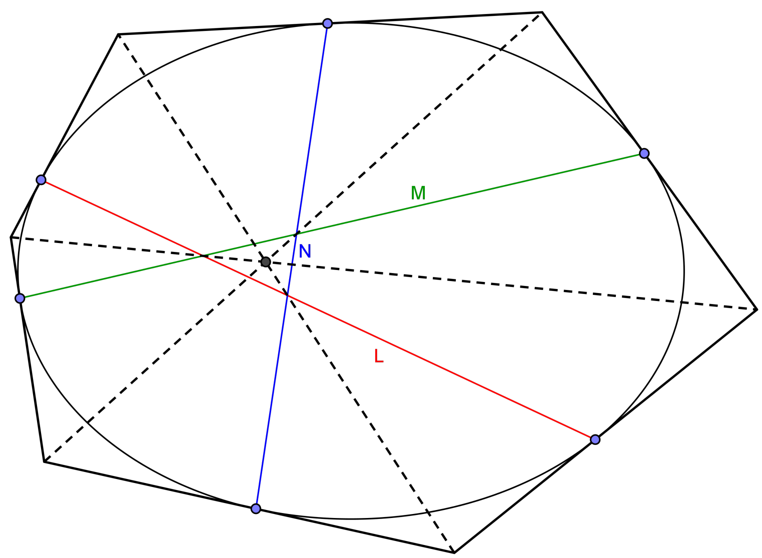


The proposition tells us that the chords of intersection between $S + L^2 = 0$ and $S + N^2 = 0$ will meet in a point with the chords L and N. But we have shown that these two conics represent four of the sides of our hexagon. Hence their chords of intersection will be the diagonals of the quadrilateral, $L \pm N = 0$, and one, $L - N = 0$, will be a diagonal of the hexagon.

Continuing in this fashion, the lemma produces the three principal diagonals, in this case the lines are

$$L - M = 0, \quad M - N = 0, \quad N - L = 0.$$

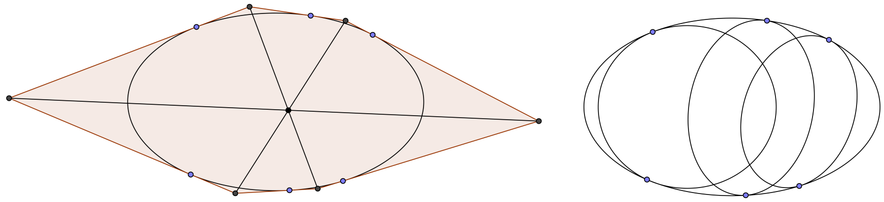
These are sure to intersect at a single point due to their linear dependence, as discussed in the lemma itself. This means that given a conic we have found the hexagon circumscribing it and shown that the principle diagonals of this hexagon must meet in a point using the abridged notation.



To complete the proof we have only to argue that for any hexagon circumscribing a conic we can take three conics each defined by a two of the six sides. The sides are, by their nature, tangent to the circumscribed conic and so we may formulate the equations in the form $S - kTT' = 0$ and proceed as we have done here. Then, by our lemma, we have shown that the principal diagonals must intersect in a point and the theorem is proved.

Notes on the Proof

In the example used to illustrate the method of proof the conics are constructed using pairs of opposite sides, however this need not be the case. Depending on which way we pair up the sides we may obtain the other sets of three lines included in the lemma. Below is an example where two of the internal conics are drawn connecting adjacent sides.



This causes the formulation of the lines to change, but as the lemma states, the result applies to all combinations and we see that we still have the expected common intersection.

Pascal's Theorem

The second classical theorem we will examine is Pascal's theorem. The theorem is named for the famous French mathematician Blaise Pascal, who formulated and proved it in 1639 at the age of 16. Pascal is also remembered in mathematics for his part in the development of both projective geometry and probability theory as new fields of study. [Salmon 267]

Theorem. *The three intersections of the opposite sides of any hexagon inscribed in a conic lie on a common, straight line.*

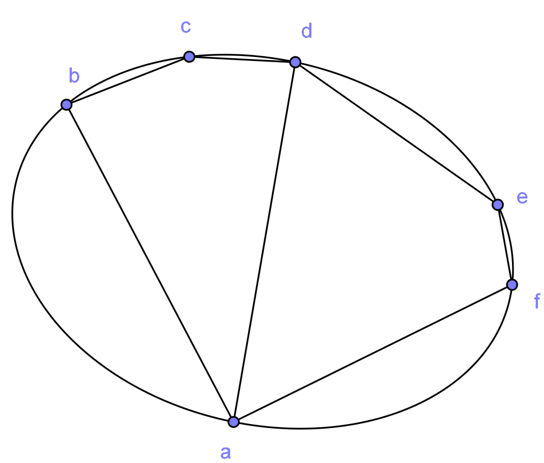
To begin, let the six vertices on a conic be named a,b,c,d,e,f and let $(ab) = 0$ denote the equation of the line connecting a and b. Then, since the conic circumscribes the quadrilateral $abcd$, the equation of the conic must be capable of being put into the form

$$(ab)(cd) - (bc)(ad) = 0.$$

But since it also circumscribes the quadrilateral $defa$, the equation must additionally be expressible in the form

$$(de)(fa) - (ef)(ad) = 0.$$

For both of the above we are using, once again, the form $S - kS' = 0$ to define a conic through four points. Here we have simply supposed k to be equal to one and that our conics are pairs of lines giving us an equation of the form $\alpha\gamma - k\beta\delta = 0$. For the equations we are taking two different sets of four points from the six vertices that lie on the conic. The following figure is an example using an ellipse.



We may equate the two expressions,

$$(ad)(cd) - (bc)(ad) = (de)(fa) - (ef)(ad),$$

and then we rearrange them to obtain

$$(ab)(cd) - (de)(fa) = [(bc) - (ef)](ad).$$

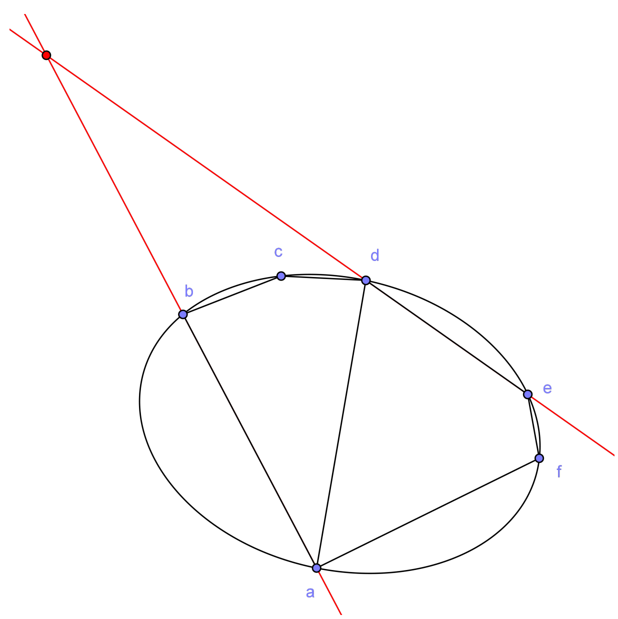
This tells us that the left-hand side of this equation, which represents a quadrilateral formed by the lines (ab), (de), (cd), (af), is resolvable into two factors on the right-hand side. These factors must therefore represent the diagonals of that quadrilateral.

The first is (ad), the line joining the points a and d. The other must be $[(bc) - (ef)]$ which joins the other two points of our quadrilateral. For us to find these two points we must examine how the points a and d arise from the left-hand side of the equation:

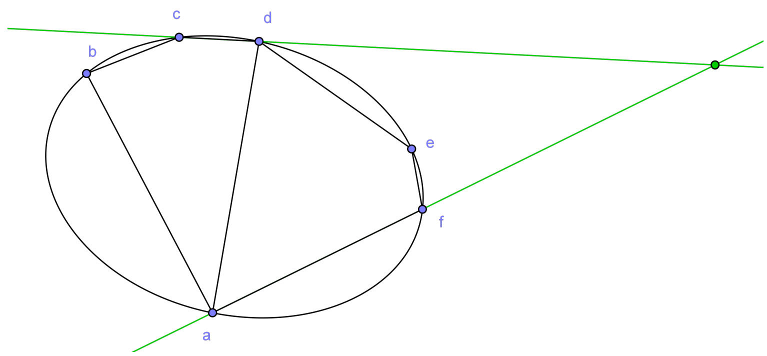
$$(ab)(cd) - (de)(fa).$$

The point a is the intersection of the lines (ab) and (fa), and d is the intersection of (cd) and (de). We have taken the intersection of one factor from the left of the minus sign and one from the right.

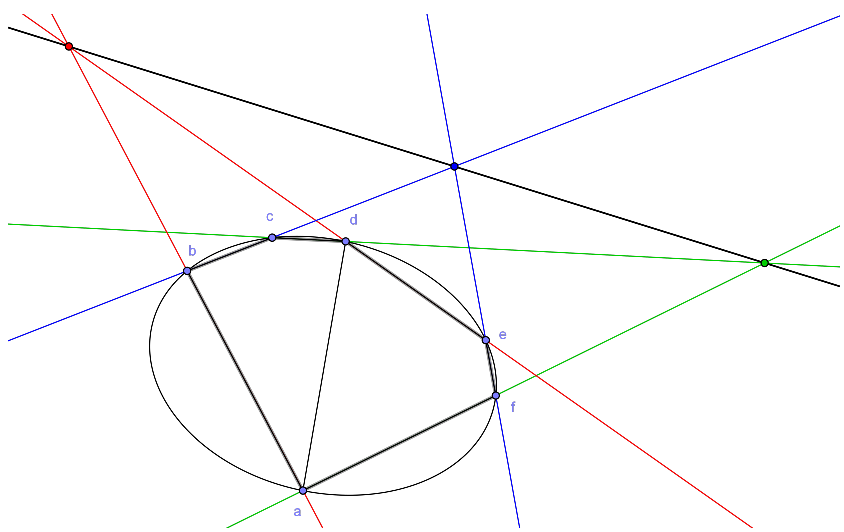
To find our other two points we then look to the intersection of (ab) with (de), pairing (ad) with its other possible partner.



Then, similarly, we take the the intersection of (fa) with (cd). Note how we are pairing opposite sides of our hexagon so the intersection points we obtain are two of the points that we wish to show as lying on a common line.



Thus we have obtained the two other points of our quadrilateral and seen that they are the intersection of two pairs of opposite sides of our hexagon. We return to consider $[(bc) - (ef)]$ as the line joining those two points and observe that this is formulated as a line through the intersection of (bc) and (ef), the two remaining opposite sides, expressed in the form $\alpha - k\beta$. Thus we have shown that all three intersections lie on one line.



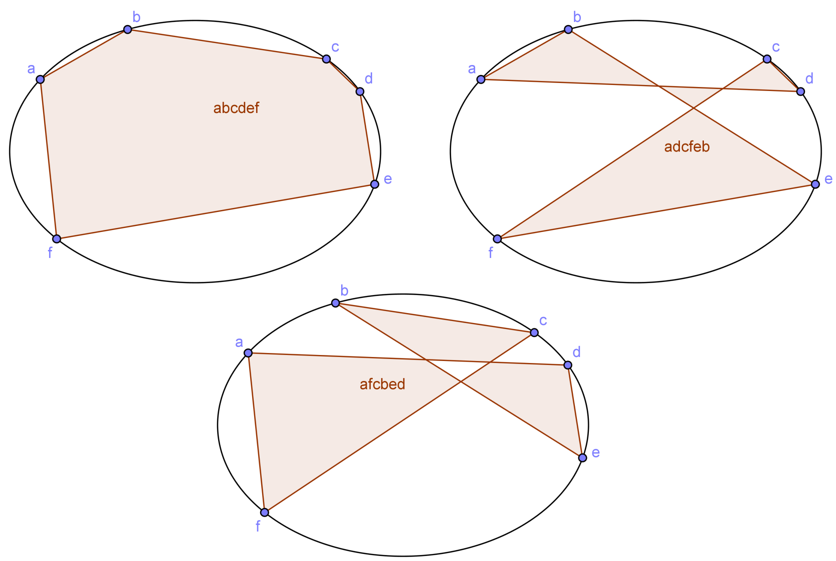
In a similar fashion to Brianchon's theorem there are several different cases on a given six points and which case we obtain depends on the different orders in which we take the points. This leads us directly into our next theorem.

Steiner's Theorem of Hexagons

This theorem is named for Jakob Steiner, the Swiss mathematician who proved it. Steiner was prolific within the field of geometry, considerably advancing it over the course of his life from 1796 - 1863. Indeed, the fourth result which we will examine is also a product of his work. [Salmon 268]

Theorem. *Given a hexagon inscribed in a conic the three Pascal lines, obtained by applying Pascal's theorem to the vertices in the orders $abcdef$, $adcfeb$ and $afcbed$ will meet in a point.*

The example used to illustrate the proof of Pascal's theorem used the vertex order $abcdef$ and we obtained a line through the intersection of the opposite vertices. The figures below show an example of the three hexagons arising from the three vertex orders stated in this theorem.



We shall now work to apply Pascal's theorem to each of these hexagons and obtain the three Pascal lines. It is important to note that while the hexagon may be different, the circumscribing conic is fixed. This means we still have the fact that the conic circumscribes the quadrilaterals $abcd$ and $defa$ as we used in the proof of Pascal's Theorem. We also note that the conic circumscribes a third quadrilateral, $bcef$.

This tells us that we may represent the conic using the equations

$$(ab)(cd) - (bc)(ad) = 0, \tag{1}$$

$$(de)(fa) - (ef)(ad) = 0, \tag{2}$$

$$(be)(cf) - (bc)(ef) = 0. \tag{3}$$

We will now break the proof into three cases, one for each hexagon. We shall carefully formulate, using the three equations above, the three Pascal lines so that we may ultimately demonstrate that they are necessarily linearly dependent and so intersect at a common point.

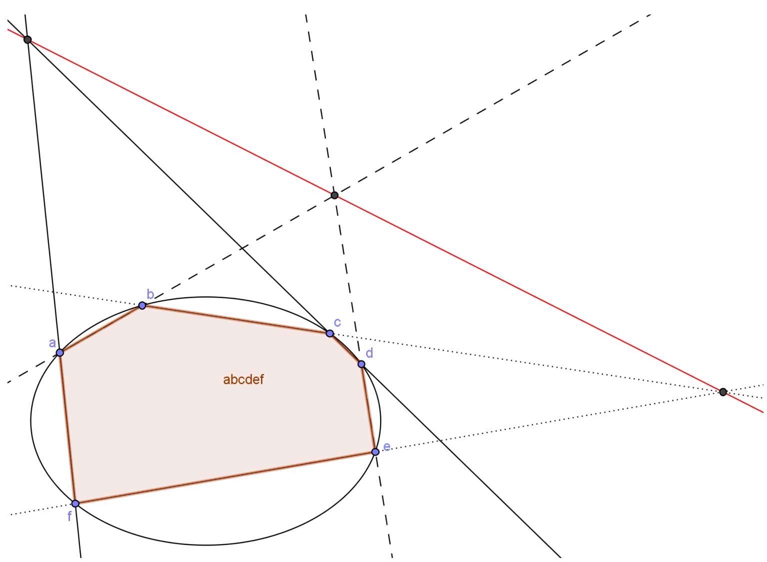
Case 1: $abcdef$

In (1) we see we have three lines which are edges of this hexagon, (ab), (cd), and (bc), plus one line which is not, (ad). Then in (2) we see we have (de), (fa) and (ef) which are edges of the hexagon and (ad) which is not. It is important that our non-edge line is common to the two equations we will choose. For (3) we see that we have only two edges (bc) and (ef) reinforcing our choice to use (1) and (2) to characterize this case.

We combine (1) and (2) to obtain

$$(ad)(cd) - (de)(fa) = [(bc) - (ef)](ad).$$

Then by Pascal's theorem the points of intersection $((bc), (ef))$, $((ab), (de))$ and $((cd), (af))$ all lie on the line $(bc) - (ef)$. Note that we have employed the notation (α, β) for the point of intersection rather than $\alpha\beta$ to avoid confusing it with the product of the two lines. We shall also use this notation in the other cases for the same reason.



The figure employs three styles of line, solid, dashed and dotted to help illustrate the different pairs. It should not be taken as an indication that the dotted lines are less important than the solid ones. The Pascal line for this case is coloured red.

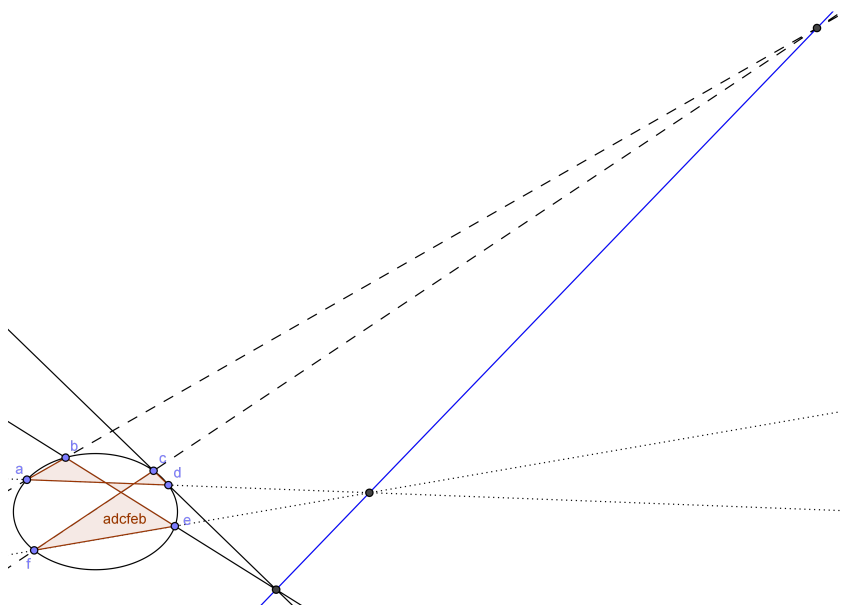
Case 2: $adcfeb$

For (1) we have edges (ad), (cd) and (ab) with the line (bc) not an edge. For (2) we have edges (ef) and (ad) only. Lastly, for (3) we have edges (be), (cf) and (ef) with the line (bc), again, not an edge. Thus we will use (1) and (3) to characterize this case.

We combine (1) and (3) to obtain

$$(ab)(cd) - (bc)(cf) = [(ad) - (ef)](bc).$$

Then by Pascal's theorem the points of intersection $((ab), (cf))$, $((cd), (be))$ and $((ad), (ef))$ all lie on the line $(ad) - (ef)$, coloured blue in the figure.



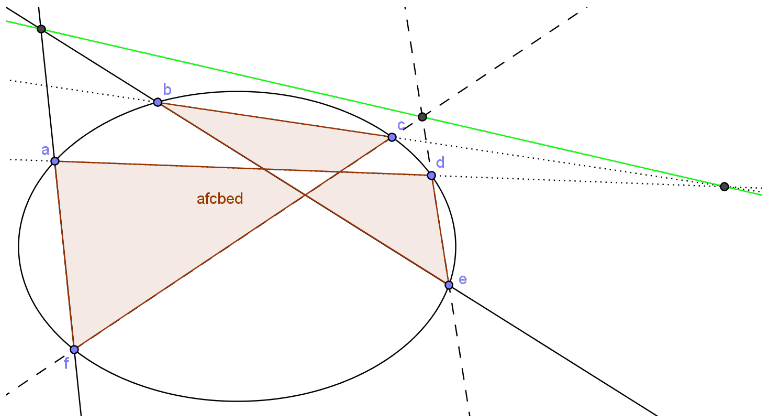
Case 3: $afcbed$

For (1) we have edges (bc) and (ad) only. For (2) we have edges (de), (fa) and (ad) with the fourth line (ef). And for (3) we have edges (be), (cf) and (bc) with, again, the line (ef) not an edge. Thus we use (2) and (3) to characterize this case.

We combine (2) and (3) to obtain

$$(de)(fa) - (be)(cf) = [(ad) - (bc)](ef).$$

Then by Pascal's theorem the points $((de), (cf))$, $((fa), (be))$ and $((ad), (bc))$ all lie on the line $(ad) - (bc)$, coloured green in the figure.

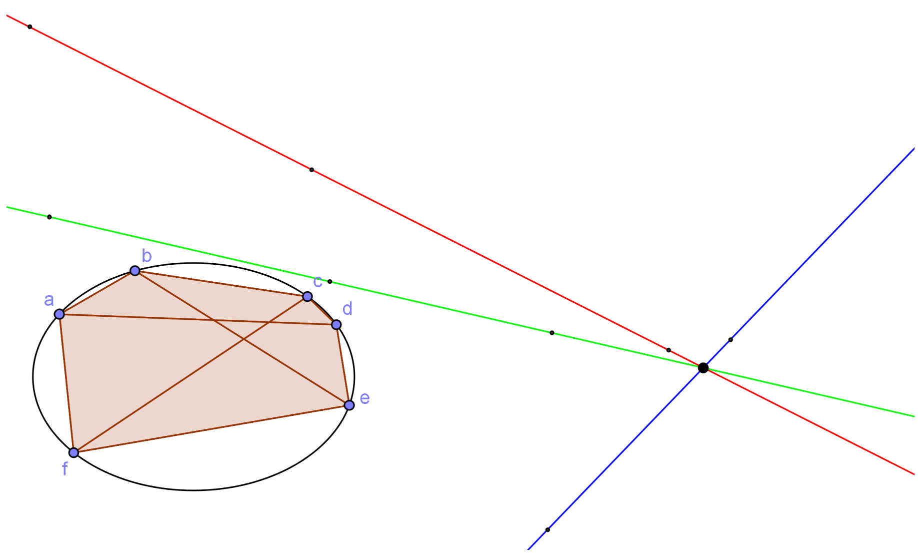


Conclusion

Thus we have obtained the three Pascal lines as

$$(bc) - (ef) = 0, \quad (ef) - (ad) = 0, \quad (ad) - (bc) = 0.$$

It is simple to see that they are linearly dependent, $[(bc) - (ef)] + [(ef) - (ad)] = [(bc) - (ad)]$, and it follows that these three lines must meet in a point as we originally claimed.



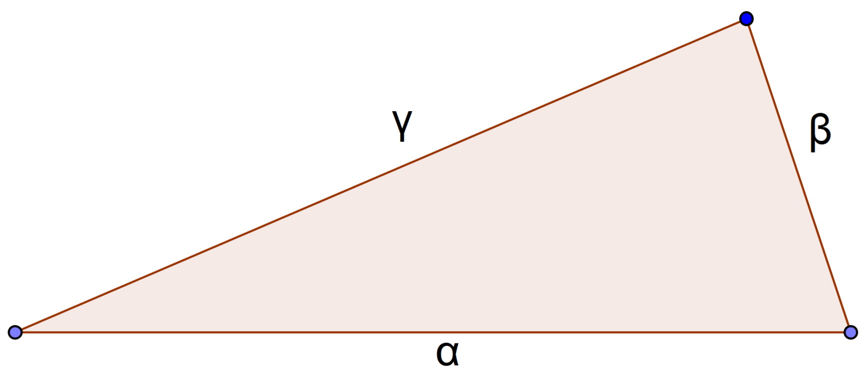
Thus the theorem is proved. In the final figure the pascal lines are displayed in their respective colours from each case and the third point on the blue line has been excluded from the figure for the sake of a better overall scale.

The Malfatti Problem

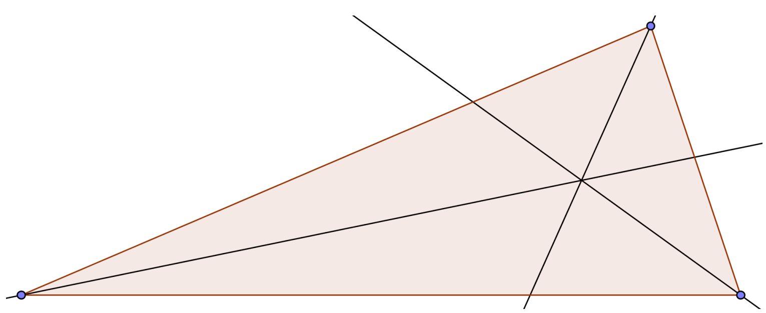
The final classical result we wish to consider is a solution to the Malfatti problem, originally posed in 1803 by the Italian mathematician Gian Francesco Malfatti. A few variations on the statement exist but we will be using the statement given below. [Salmon 288]

Problem. *How may we inscribe an arbitrary triangle with three circles, each circle touching two sides and all circles touching each other?*

We will demonstrate a solution by Steiner, using the abridged notation we have developed, to build a method of constructing the Malfatti circles for any triangle. We take three lines forming a triangle $\alpha = 0$, $\beta = 0$ and $\gamma = 0$.



We first observe that the centers of the Malfatti circles must lie on the angle bisectors of the triangle in order to be equidistant from the two edges. These bisectors are easily formulated as $\alpha - \beta$, $\beta - \gamma$ and $\gamma - \alpha$ as we have done before in our notation. These are lines through the intersection of two sides, which are exactly the vertices, with $k = 1$ to give bisectors of the angles of the triangle. We further note that these three are linearly dependent and therefore meet in a point which we call the *incenter*.

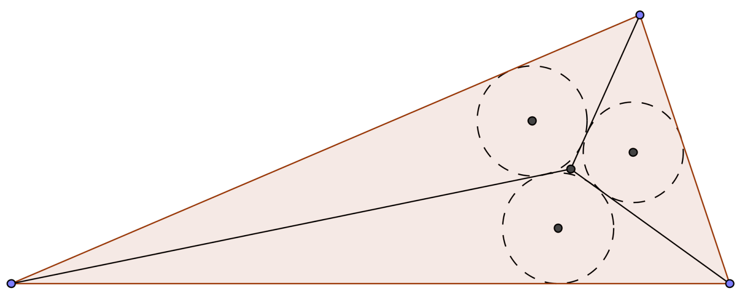


We use this to partition our triangles into three lesser triangles and we then inscribe circles into of each of these sub triangles such that they touch each of the sides. The equation of such a circle, as discussed in an earlier section, must take the form

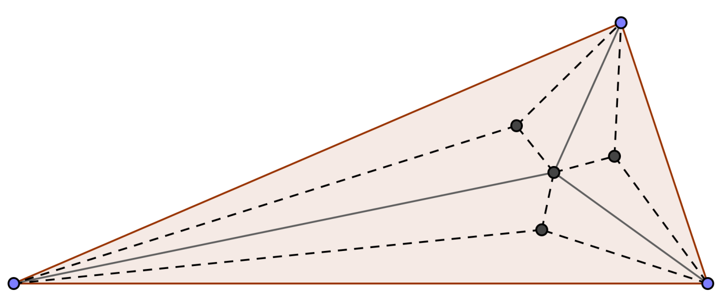
$$\cos\frac{1}{2}A\sqrt{\alpha'} + \cos\frac{1}{2}B\sqrt{\beta'} + \cos\frac{1}{2}C\sqrt{\gamma'} = 0.$$

where $\alpha' = 0$, $\beta' = 0$ and $\gamma' = 0$ are the lines forming the sides of the triangle and A, B and C are the angles.

We will use C_α, C_β and C_γ to denote the inscribed circles of the lesser triangles, where C_α is the lesser triangle with $\alpha = 0$ as one of its sides and following the same convention for C_β and C_γ.



It is worth addressing the geometric approach to constructing these circles as an aside. For this we would find the incenter, which is the center of the inscribed circle, of each by looking at the intersection of the angle bisectors as before. Then the radius of the circle can be seen to be equal to the perpendicular distance from the center to the outer edge of the triangle. Thus we have the center and radius of the circles.

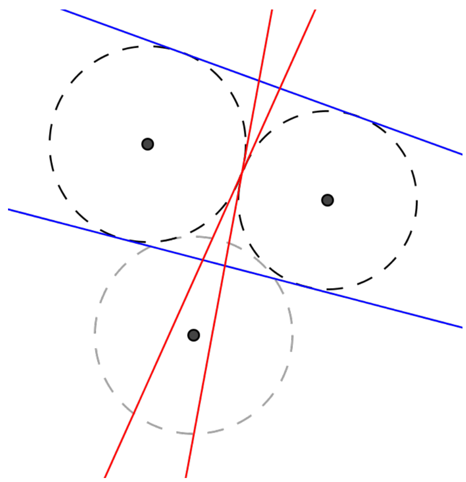


From here the key step in the construction is to take the tangents that touch two circles and meet in a point.

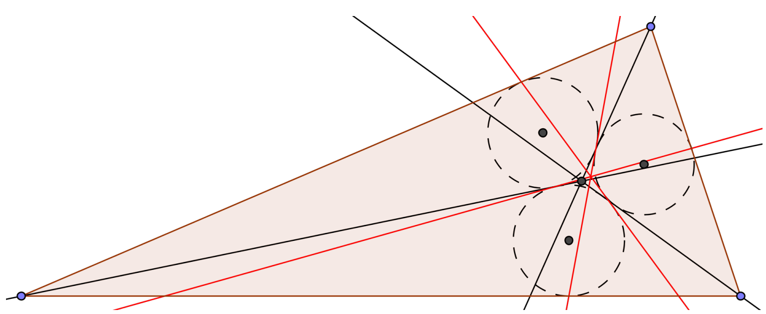
To do this we consider things a different way, instead of a line per se, we consider a conic having double contact with two circles. This means that our conic, which we shall call μ, is in the form of two lines and each is tangent to both the circles. The equation of such a conic μ_{ij} is, with i, j any two distinct choices from α, β, γ

$$(\mu_{ij})^2 - 2\mu_{ij}(C_i + C_j) + (C_i - C_j)^2 = 0.$$

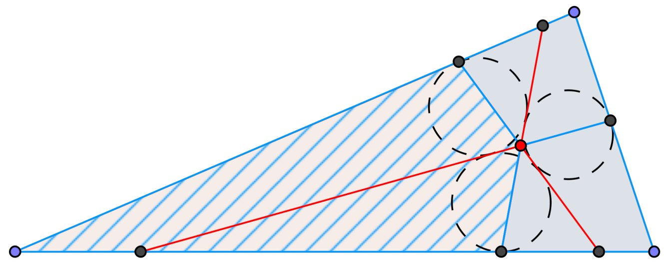
This is shown below for one pair of inscribed circles, C_β and C_γ,



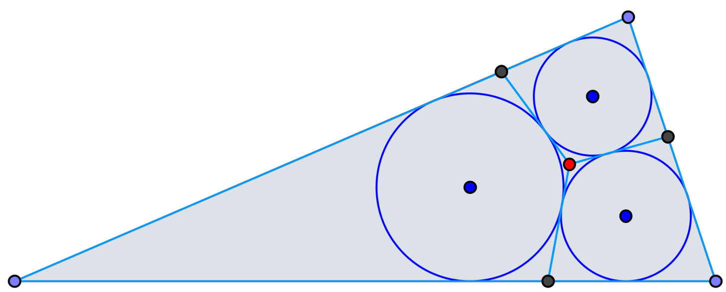
We see there are two possible conics, shown above in blue and red, however the blue conics will not meet in a point if we take them for the other pairs of circles. Hence we take the red conic to be $\mu_{\beta\gamma}$ and we name its individual lines L_1 and L_2. Similarly we take $\mu_{\alpha\beta}$ to split into M_1 and M_2, and $\mu_{\gamma\alpha}$ to split into N_1 and N_2.



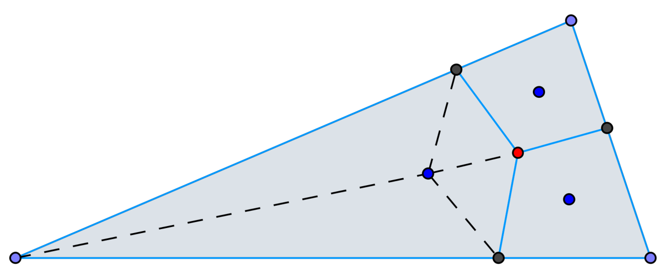
When we show all three conics $\mu_{\alpha\beta}$, $\mu_{\beta\gamma}$ and $\mu_{\gamma\alpha}$ on the triangle we see that they all three meet in two points. When we factor them each into their two lines we see one line from each conic passes through the intersections. Shown in black we see the lines L_2, M_2 and N_2 intersect in a common point and lie exactly as the angle bisectors of our triangle, something we had constructed before. However the lines L_1, M_1 and N_1 shown in red and meeting in a common point are not a construction we have seen before.



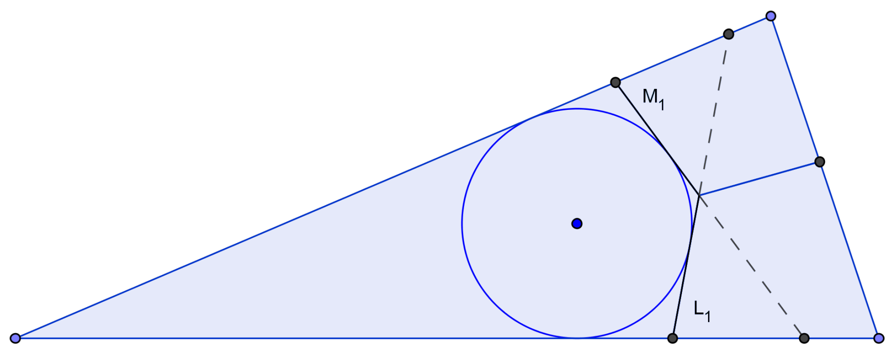
We use the lines L_1, M_1 and N_1 to partition our triangle in a new way, forming three quadrilaterals. Further, these quadrilaterals can be inscribed with a circle and the inscribed circles of these three quadrilaterals are exactly the Malfatti circles of our triangle.



As with triangles we can geometrically construct the inscribed circle of an inscribable quadrilateral by finding the incenter; finding the intersection of the angle bisectors and taking the perpendicular distance from the incenter to an outer edge of the triangle.



In the abridged notation we also refer back to the case of the triangle. Looking at the figure below it is plain to see that the inscribed circle of the quadrilateral formed by $\alpha = 0$, $\gamma = 0$, $L_1 = 0$ and $M_1 = 0$ is equally the inscribed circle of the triangle formed by $\alpha = 0$, $\gamma = 0$ and either $L_1 = 0$ or $M_1 = 0$.



Thus, we may construct it using the same formulation as we used previously, with appropriately chosen lines α', β', γ' and angles A, B, C

$$\cos\frac{1}{2}A\sqrt{\alpha'} + \cos\frac{1}{2}B\sqrt{\beta'} + \cos\frac{1}{2}C\sqrt{\gamma'} = 0.$$

Similarly for the other Malfatti circles we may choose arbitrarily one of the two lines that are part of the tangent conics we called μ_{ij} and consider the Malfatti circle as the inscribed circle of a triangle. This triangle will always be formed by two of the outer sides and the one chosen line from a conic μ_{ij}.

Hence we have constructed the Malfatti circles of our triangle using Steiner's method and Salmon's abridged notation.

Notes on the construction

This construction shies away from chasing the notation into its messiest details because to do so would serve only to bloat a simple idea. In particular we refrain from any example where we substitute values into the formula of an inscribed circle in order to keep things simple. This is the reason behind including the geometric construction of those inscribed circles, to underline the simplicity of those objects.

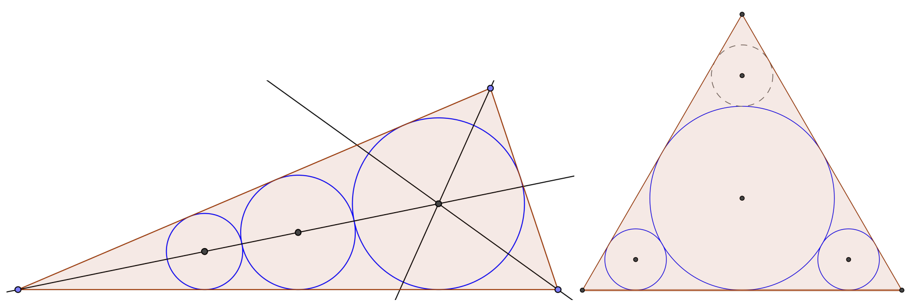
The Volumetric Malfatti Problem

There is one more historical note worth making about the Malfatti problem. As mentioned there are other statements and here we will briefly consider the main, non-equivalent alternative.

When Malfatti first presented the problem he asked *how may three cylindrical columns of marble be most efficiently cut, with respect to maximizing volume, from a triangular prism of marble.*

He assumed, as many others did, that the answer to this question lay in the Malfatti circles and so his problem, as it were, became how to construct those circles. We have just seen Steiner's solution to the construction of the Malfatti circles, presented using the abridged notation we have developed.

However the conjecture that the Malfatti circles maximized the volume of the hypothetical cylinders was later proved to be incorrect. In fact Goldberg proved in 1967 that the Malfatti circles are never optimal in terms of volume, instead a "greedy" procedure devised by Lob and Richmond in 1930 will maximize the total volume. We will here briefly show the solution to this original, volumetric Malfatti problem.



In brief, this procedure is simply to construct the largest possible circle each time, this is why the procedure is called greedy. This will always choose the inscribed circle first and then the circle inscribed inside the most acute angle and touching the first circle. The third circle may then lie either in the same corner as the second or either of the others, we choose the largest and in the case of two equal options we choose arbitrarily.

The figures show the result of this procedure, first for the scalene triangle we used in the construction of the Malfatti circles and also for an equilateral triangle. Since in the equilateral case the possible second circles, between the inscribed circle and a corner, are identical we could choose any two of them and so the third is also outlined.

This section is intended to show some of the historical context for the problem we have looked at, however we are not concerned in this document with further investigations or details of the Malfatti problem.

Conclusion

The abridged notation began with a formulation of lines in terms of perpendicular distance and a convention of abbreviating such lines. From there we have worked to build the notation until it is no longer necessarily dependent on Cartesian considerations. Subsequently we have worked to see how it can be used to consider polygons and conics in a complex way and also used it to prove four classical results in the latter part of this thesis.

We have worked to gain a deep understanding of this notation, its use and its advantages over other systems. Salmon's abridged notation does not have the level of precision that a Cartesian coordinate system offers, but its simplicity has allowed us to present ideas with an appealing clarity. In complex Cartesian systems a simple idea can become entangled in a sprawl of notation that obscures some of the intuitive beauty of geometry.

We have, by the end of this thesis, developed Salmon's abridged notation to a level where it could now be readily applied to a wide variety of other results in geometry. It presents an interesting perspective that falls half way between a coordinate free, synthetic geometry and the strictly coordinate based Cartesian geometry. This demonstrates in general that the most common way to view mathematical problems and systems is not necessarily the only way or the best one.

Reference

[Salmon] George Salmon *A Treatise on Conic Sections Fifth Edition: Longman, Green, Reader, and Dyer* 1869.